On superalgebras

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Abstract

In this article we introduce the definition of associative superalgebras, basic characteristics, and give some examples.

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1 Introduction

In the last few decades one of the most active and fertile subjects in algebra is recently developed theory of graded algebras and so called superalgebras. In [12] Kac wrote that the interest of the field of superalgebras appeared in physic in the contest of "supersimetry". A lot of results about superalgebras and graded algebras has been written by Kac, Martinez, Zelmanov, Wall, Shestakov and others (see for example [7, 8, 9, 11, 12, 13, 14, 15]). The main goal of this paper is to introduce a definition of associative superalgebras, give some examples, and present some basic properties.

By an algebra we shall mean an associative algebra over the field Φ. We will assume that the definitions of algebra, module, and ideal is well known. However we shall write some definitions and explain some basic properties of an algebra. An algebra $\mathcal{A}$ is *simple*, if $\mathcal{A}^2 \neq 0$ and the only ideals of $\mathcal{A}$ are 0 and $\mathcal{A}$. We say that an algebra $\mathcal{A}$ is *prime* if the product of two nonzero ideals is nonzero. This is equivalent to the following implication: if $a\mathcal{A}b = 0$

for some $a, b \in \mathcal{A}$, it follows that either $a = 0$ or $b = 0$. The example of a prime algebra is $M_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices. The algebra is called *semiprime* if it has no nonzero nilpotent ideals (an ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is called *nilpotent*, if $\mathcal{I}^n = 0$ for some number $n \in \mathbb{N}$). This is equivalent to the property that $a\mathcal{A}a = 0$ for some $a \in \mathcal{A}$ implies that $a = 0$. Every prime algebra is semiprime algebra. It turns out that the converse is in general not true. Namely, if $0 \neq \mathcal{A}$ is prime algebra, then $\mathcal{A} \times \mathcal{A}$ is semiprime algebra, which is not prime.

2 Superalgebras

Dear readers, in the following chapter we invite you to the world of superalgebras. We will introduce some basic definitions and present some examples of associative superalgebras.

A *superalgebra* is a $\mathbb{Z}_2$-graded algebra. This means that there exist Φ-submodules $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{A}_0\mathcal{A}_0 \subseteq \mathcal{A}_0$ (that means $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$), $\mathcal{A}_0\mathcal{A}_1 \subseteq \mathcal{A}_1, \mathcal{A}_1\mathcal{A}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A}_1\mathcal{A}_1 \subseteq \mathcal{A}_0$. We say that $\mathcal{A}_0$ is the even, and $\mathcal{A}_1$ is the odd part of $\mathcal{A}$.

An associative superalgebra $\mathcal{A}$ is an associative $\mathbb{Z}_2$-graded algebra. We say that $\mathcal{A}$ is a *trivial superalgebra*, if $\mathcal{A}_1 = 0$. If $a \in \mathcal{A}_k$, $k = 0$ or $k = 1$, then we say that a is *homogeneous of degree* k and we write $|a| = k$.

A *graded* Φ*-submodule* $\mathcal{B}$ of an associative superalgebra $\mathcal{A}$ is such submodule of an algebra $\mathcal{A}$ that

$$\mathcal{B} = \mathcal{B} \cap \mathcal{A}_0 \oplus \mathcal{B} \cap \mathcal{A}_1.$$

In this case we write $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{A}_0$ and $\mathcal{B}_1 = \mathcal{B} \cap \mathcal{A}_1$. That means $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$. If $\mathcal{B}$ is a graded subalgebra of $\mathcal{A}$, than $\mathcal{B}$ is also an associative superalgebra. A *graded ideal* (or superideal) $\mathcal{I}$ of a superalgebra $\mathcal{A}$ is an ideal of $\mathcal{A}$, which is also a graded Φ-submodule. That is $\mathcal{I} = \mathcal{I} \cap \mathcal{A}_0 \oplus \mathcal{I} \cap \mathcal{A}_1$ or $\mathcal{I} = \mathcal{I}_0 \oplus \mathcal{I}_1$.

Let us write something about the gradation. The natural question is how to make a decision about $\mathbb{Z}_2$-gradation? Given an associative superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, we define $\sigma : \mathcal{A} \to \mathcal{A}$ by $(a_0 + a_1)^\sigma = a_0 - a_1$. Note that σ is an automorphism of $\mathcal{A}$ such that $\sigma^2 = id$. Conversely, given an algebra $\mathcal{A}$ and an automorphism σ of $\mathcal{A}$ with $\sigma^2 = id$, $\mathcal{A}$ then becomes a superalgebra by defining $\mathcal{A}_0 = \{a \in \mathcal{A} \mid \sigma(a) = a\}$ and $\mathcal{A}_1 = \{a \in \mathcal{A} \mid \sigma(a) = -a\}$ (indeed, any element $a \in \mathcal{A}$ can be written as $a = \frac{a+a^\sigma}{2} + \frac{a-a^\sigma}{2}$ and $\frac{a+a^\sigma}{2} \in \mathcal{A}_0$, $\frac{a-a^\sigma}{2} \in \mathcal{A}_1$). That is to say, the $\mathbb{Z}_2$-grading can be characterized via the automorphism with square id.

A submodule $\mathcal{B}$ of a superalgebra $\mathcal{A}$ is graded if and only if $\mathcal{B}^\sigma = \mathcal{B}$. Let the center $\mathcal{Z}(\mathcal{A})$ of a superalgebra $\mathcal{A}$ be the usual center of an algebra $\mathcal{A}$, that is $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} \mid ab = ba \; \forall b \in \mathcal{A}\}$. The center is graded, since automorphism maps the center into itself. That means $\mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\mathcal{A})_0 \oplus \mathcal{Z}(\mathcal{A})_1$.

In what follows we shall present some examples of associative superalgebras.

Example 2.1 Let $\mathcal{A}$ be an algebra and let $c \in \mathcal{A}$ be an invertible element. Further, let σ be an automorphism of an algebra $\mathcal{A}$, which is defined by $x^\sigma = cxc^{-1}$ for all $x \in \mathcal{A}$. We see that $\sigma^2 = id$ if and only if $c^2 \in \mathcal{Z}(\mathcal{A})$. It follows that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a superalgebra, where $\mathcal{A}_0 = \{x \in \mathcal{A} \mid xc = cx\}$ and $\mathcal{A}_1 = \{x \in \mathcal{A} \mid xc = -cx\}$.

In particular, let $\mathcal{A} = M_{r+s}(\Phi)$ be an algebra of all $(r+s) \times (r+s)$ matrices over Φ, $r, s \in \mathbb{N}$. For the element c we can choose a matrix $\begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix}$, where I_r is an identity matrix of $M_r(\Phi)$ and I_s is an identity matrix of $M_s(\Phi)$. Then the even and odd parts are given by

$$\mathcal{A}_0 = \begin{bmatrix} M_r(\Phi) & 0 \\ 0 & M_s(\Phi) \end{bmatrix} \quad \text{in } \mathcal{A}_1 = \begin{bmatrix} 0 & M_{r,s}(\Phi) \\ M_{s,r}(\Phi) & 0 \end{bmatrix},$$

where $M_{r,s}(\Phi)$ is the set of $r \times s$ matrices. This algebra is an associative superalgebra and it is usually written as $M(r|s)$.

Example 2.2 Let A be an algebra over Φ and let $\mathcal{A} = A \times A$. Furthermore, let σ be an automorphism on $\mathcal{A}$, defined by $\sigma(a,b) = (b,a)$, $a, b \in A$. Then we have $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, where the even part is written in the form $\mathcal{A}_0 = \{(a,a) \mid a \in A\}$ and the odd part in the form $\mathcal{A}_1 = \{(b,-b) \mid b \in A\}$. It turns out that $\mathcal{A} \cong \{\begin{bmatrix} C & D \\ D & C \end{bmatrix} \mid \; C, D \in A\}$,

$$\mathcal{A}_0 \cong \{\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \mid \; C \in A\} \quad \text{and} \quad \mathcal{A}_1 \cong \{\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} \mid \; D \in A\}.$$

In this case we say that the superalgebra $\mathcal{A}$ is given by the exchange automorphism.

Example 2.3 Let $\mathcal{A} = Q(\alpha, \beta)$ be a 4-dimensional algebra over Φ with a base $\{1, uv, u, v\}$ and let the multiplication be defined as follows $u^2 = \alpha \in \Phi$, $v^2 = \beta \in \Phi$, $uv = -vu$. In particular $\mathcal{A}$ is the algebra of quaternions over $\mathbb{R}$. Let us write $\mathcal{A}_0 = \Phi 1 + \Phi uv$ and $\mathcal{A}_1 = \Phi u + \Phi v$. Then it follows that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is an associative superalgebra which is called the superalgebra of quaternions.

Let us write some basic properties of associative superalgebras. An associative superalgebra $\mathcal{A}$ is *simple*, if it has no proper nonzero graded ideals. The only graded ideals are 0 and the whole superalgebra $\mathcal{A}$. Note that this does not mean that the simple superalgebra is simple as an algebra. If the product of two nonzero graded ideals of a superalgebra $\mathcal{A}$ is nonzero, the superalgebra $\mathcal{A}$ is called a *prime-superalgebra*. The superalgebra $\mathcal{A}$ is a *semiprime-superalgebra*, if it has no nonzero nilpotent graded ideals. As noted in [1], this is equivalent to the condition that $a\mathcal{A}b = 0$, where a and b are any *homogeneous elements* in $\mathcal{A}$, implies $a = 0$ or $b = 0$. In fact, the same conclusion holds true if we assume that only one of these two elements, say b, is homogeneous.

Let $\mathcal{A}$ be a prime-superalgebra. The natural question that appears is: are the algebras $\mathcal{A}$ in $\mathcal{A}_0$ prime algebras as well? The next two examples show that this is not always true.

Example 2.4 Let A be a prime algebra over Φ and let $\mathcal{A} = A \times A$ be a superalgebra with gradation defined as in the example 2.2. This algebra is a prime-superalgebra (the product of any two nonzero graded ideals is nonzero), which is not a prime algebra, since $(0 \times A)(A \times 0) = 0$.

Example 2.5 The superalgebra $M(r|s)$ is a prime-superalgebra. The sets

$$\mathcal{I} = \{\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \mid C \in M_r(\mathbb{F})\} \quad \text{and} \quad \mathcal{J} = \{\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \mid D \in M_s(\mathbb{F})\}$$

are nonzero ideals of an algebra $M(r|s)_0$ such that the product of them is zero. Thus, the algebra $M(r|s)_0$ is not a prime algebra.

The answer about the connection between prime-superalgebra (or semiprime-superalgebra) $\mathcal{A}$ and prime algebras (or semiprime algebras) $\mathcal{A}$ and $\mathcal{A}_0$ is: if $\mathcal{A}$ is an associative semiprime-superalgebra, then $\mathcal{A}$ and $\mathcal{A}_0$ are also semiprime algebras. In case $\mathcal{A}$ is an associative prime-superalgebra, then either $\mathcal{A}$ is prime algebra or $\mathcal{A}_0$ is prime algebra. The proof of those results we can find in [13].

3 Conclusion

The natural question that appears is how to generalize some classical structures. Let us briefly describe the background. For example: let $\mathcal{A}$ be an associative algebra. Introducing a new product in $\mathcal{A}$, the so called Jordan product $a \circ b = ab + ba$, $\mathcal{A}$ becomes a Jordan algebra, usually written as $\mathcal{A}^+$. The question is what is the connection between the structural properties of algebras $\mathcal{A}$ and $\mathcal{A}^+$ (for example, every ideal of an algebra $\mathcal{A}$ is an ideal of $\mathcal{A}^+$, is the converse true?). Such questions were considered by Herstein in the 1950's (see [10]). He considered mainly simple algebras. Lately his theory was generalized. On this field a lot of papers were written by Lanski, Martindale, McCrimmon, Miers, Montgomery and many others. In the similar way we can introduce Jordan superalgebras. Again, the natural question is: what is the connection between the structure of superalgebras and Jordan superalgebras? We refer the reader to see for example [1, 2, 3, 4, 5, 7, 8, 9, 13].

At the end let us write that we can extent superalgebras to $\mathcal{G}$-graded algebras, where $\mathcal{G}$ is an Abelian group. An algebra is $\mathcal{G}$-graded, if there exist subspaces $\mathcal{A}_g$, $g \in \mathcal{G}$, of $\mathcal{A}$, such that $\mathcal{A} = \oplus_{g\in\mathcal{G}}\mathcal{A}_g$ and $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in \mathcal{G}$. Superagebras are actually a special case of $\mathcal{G}$-graded algebras. In that case $\mathcal{G} = \mathbb{Z}_2$. In the field of $\mathcal{G}$-graded algebras we can also define structures such as modules, ideals, graded prime algebras, ... And therefore new natural problems appear.

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