GRAPH THEORY AND LOGISTICS

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Abstract

This article aims to deal with logistics and theory of graphs. We will describe the connection by the real-life logistics problems and graph theory.

Key words: graph theory, logistics

1 Introduction

The word logistics comes up in different mediums, different papers or magazines. What is logistics? Logistics is the management of the flow of goods, information and other resources including energy and people, between the point of origin and the point of consumption in order to meet the requirements of consumers (frequently, and originally, military organizations). Logistics involve the integration of information, transportation, inventory, warehousing, material-handling, and packaging.

There are many practical real-life logistics problems. For example: mail delivery, garbage collection, the snow plugging problem, salt gritting on icy roads during the winter, street cleaning, street inspection, road maintenance, school bus routing and many others. Winter road maintenance operations require many complex strategic and operational planning decisions. The main problems include locating depots, designing sectors, routing service vehicles, vehicle scheduling... Plowing and road scattering in winter is very important and expensive service. If the roads are not ploughed on time, or icy roads are not scattered, a lot of participants in traffic are exposed to great danger and there is also the aspect of dissatisfaction of people. Optimization of plowing and scattering has to be considered from two different points of view: security and economy. Security aspect demands most exposed spots of the net, it means spots which first become icy to be scattered first. Economy aspect demands the route of scattering to be the cheapest. All those problems can be solved by graph theory algorithms.

2 A graph theory

In mathematics and computer science a graph theory is the study of graphs: mathematical structures used to model pairwise relations between objects from a certain collection. Informally speaking a graph is a set of objects called vertices, points or nodes connected by links called lines or edges. Given a set of items called nodes and a second set of items called edges, a graph is defined as a relationship between such sets: each edge joins a pair of nodes. Graphs are represented graphically by drawing a dot for every vertex, and drawing an arc between two vertices if they are connected by an edge. If the graph is directed, the direction is indicated by drawing an arrow.

Every edge we can give a real value which means that a graph is extended with weight function. In the case, when a graph presents a net of roads, the weight function is a length of every road. Such a graph is called a weighted graph.

A graph theory has broad applications in many disciplines. With structures, which are considered by a graph theory we can model a lot of real-life problems. The beginning of the graph theory appeared from the real-life problem, which can be called today a logistics problem. This problem was formuled and solved in 1736 by the eminent Swiss mathematician Leonhard Euler. A few years later Euler published a paper *Solutio problematis ad geometriam situs pertinentis* in the magazine *Commentarii academiae scientiarum Petropolitanae*, in which he showed a formulation and a solution of the problem of *Seven bridges of Königsberg.* This article published in 1741 is now regarded as the first written paper of mathematical discipline called graph theory.

2.1 Some definitions

A graph G is a mathematical structure used to model pairwise relations between objects from a certain collection. A graph in this context refers to a nonempty set of vertices and a collection of edges that connect pairs of vertices. The set of vertices is usually denoted by $V(G)$ and the set of edges by $E(G)$. The edges can be directed or undirected, it depends on the example. A graph with all directed edges is called directed graph, otherwise it is called undirected. In a proper graph, which is by default undirected, a line from point u to point v is considered to be the same thing as a line from point v to point u. In a digraph, short for a directed graph, the two directions are counted as being distinct arcs or directed edges. If a graph G is undirected then there is no distinction between the two vertices associated with each edge. In a directed graph its edges may be directed from one vertex to another. Figures 1 and 2 are examples of directed graphs.

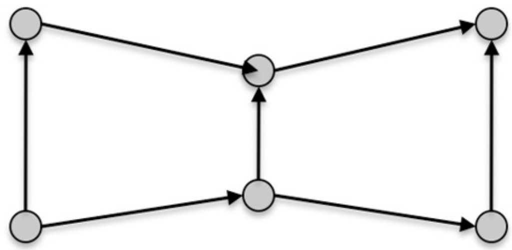

Figure 1: A directed graph with six vertices and seven edges.

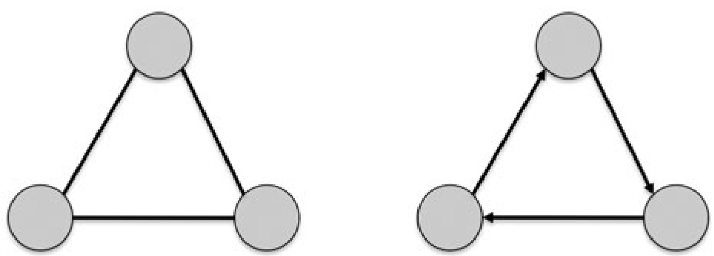

Figure 2: Undirected and directed graph.

The edge which starts and ends in the same vertex is called a loop. An edge is multiple if there is another edge with the same endvertices, otherwise it is simple (Figure 3).

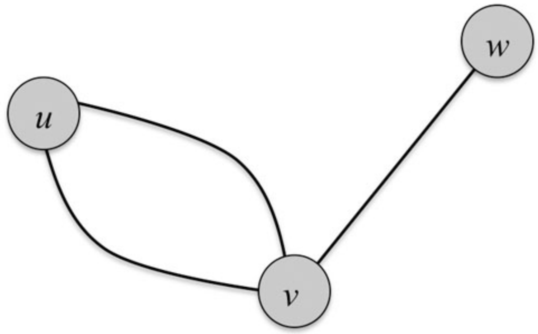

Figure 3: A multiple edge between vertices u and v.

A graph is called a simple graph if it is undirected and has no loops and no more than one edge between any two different vertices. In a simple graph each edge is a pair of distinct vertices. An edge connects two vertices. These two vertices are said to be incident to that edge or equivalently that edge incident to those two vertices. The degree of a vertex v in a graph G is the number of edges incident to v, with loops being counted twice. Figure 4 shows us that edges vu, vt , vw are incident to v. Therefore a degree of a vertex v is 3.

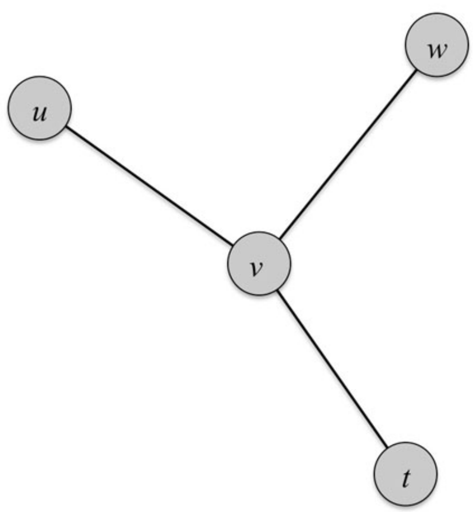

Figure 4: A degree of a vertex v is 3.

If the set of edges $E(G)$ is finite, then the total sum of vertex degrees is equal to twice the number of edges. The total degree of a graph is equal to two times the number of edges, loops included. This means that for a graph with four vertices with each vertex having a degree of two the total degree would be eight. The general formula for this is: total degree is $2n$, where n is the number of edges. Two vertices u and v are called adjacent if an edge exists between them.

Let G be a graph with the set of vertices $V(G)$ and the set of edges $E(G)$ and let G' be a graph with the set of vertices $V(G')$ and the set of edges $E(G')$. Then G' is a subgraph of a graph G if $V(G')$ is the subset of $V(G)$ and $E(G')$ is the subset of $E(G)$. A subgraph G' is a spanning subgraph of a graph G if it has the same vertex set as G. On Figure 5 we see a graph (on the right side) and its subgraph (on the left side).

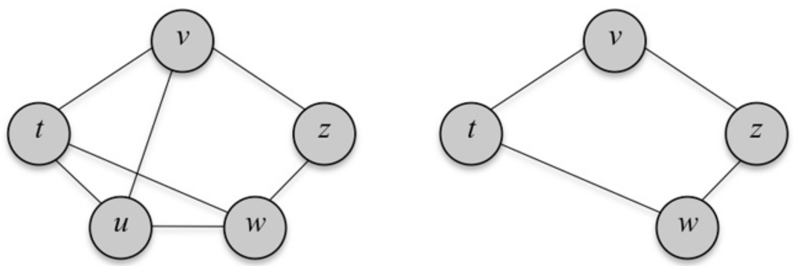

Figure 5: A graph and subgraph.

2.2 A walk and a length in a graph

A walk is an alternating sequence of vertices and edges, beginning and ending with a vertex in which each vertex is incident to the two edges that precede and follow it in the sequence, and the vertices that precede and follow an edge are the end vertices of that edge. A sequence of i edges v_0v_1, v_1v_2, …, $v_{i-1}v_i$ in a graph G is a walk from a vertex v_0 to a vertex v_i with a length i in G. A walk is usually written as $v_0v_1v_1v_2 \dots v_{i-1}v_i$. A walk is closed if its first and last vertices are the same and is open if they are different. A path is an open walk. It is usually to be simple, meaning that no vertices (and thus no edges) are repeated. Figure 6 is an example of a path.

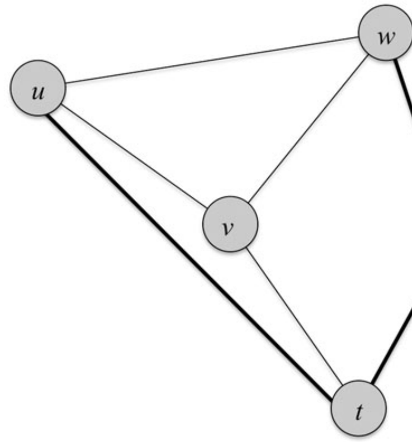

Figure 6: A walk $utzw$ is a path between vertices u and w.

A trail is a walk in which all the edges are distinct. A closed trail has been called a tour. A trail is Eulerian if it uses all edges precisely once. The distance between two vertices v_0 and v_i in a graph G is the length of a walk between them or is the number of edges that it uses. We usually denote the length by $d_G(u_0, v_i)$ (Figure 7).

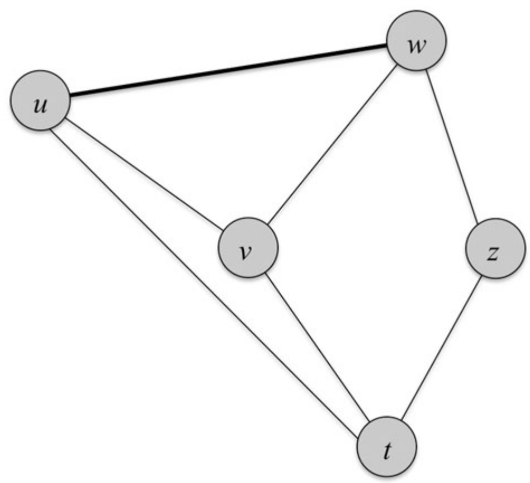

Figure 7: The distance between vertices u and v is the shortest lenght of the path between them.

The graph is said to be connected if it is possible to establish a path from any vertex to any other vertex of a graph. Otherwise the graph is disconnected. Figure 8 showes us an example of a connected graph and an example of a disconnected graph.

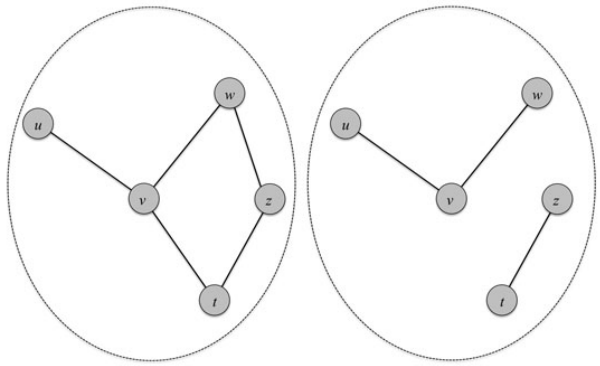

Figure 8: A connected and disconnected graph.

A graph is called a tree if any two vertices in a given graph are connected by exactly one path. Any connected graph with no cycles is a tree. Let G be a connected, undirected graph. A spanning tree of that graph is a subgraph which is a tree and connects all the vertices together. A single graph can have many different spanning trees. We can also give a weight to each edge. A minimum spanning tree is a spanning tree with weight less than or equal to the weight of every other spanning tree (Figure 9).

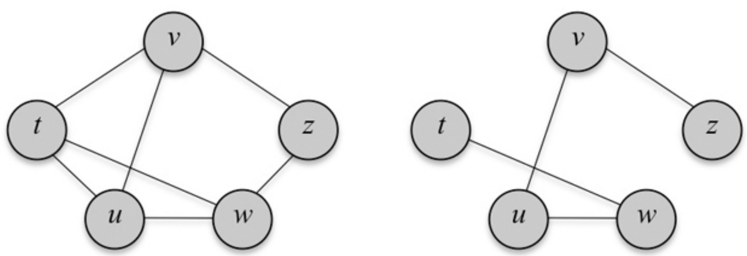

Figure 9: A graph and a spanning tree.

2.3 The problem of seven bridges of Königsberg

In this section we will present the problem of Seven bridges of Königsberg. The city of Königsberg (nowadays called Kaliningrad, administrative a part of Russia, but geographically between Poland and Lithuania) lies on the banks of river Pregel. As it is seen on the Figure 10 there are two bigger islands on the river and they are connected between them and the rest of the city with seven bridges.

The problem Euler pointed out is as following:

Does there exist a walk across all seven bridges that connects two islands in the river Pregel with the rest of the city of Konigsberg on the adjacent shores, a walk that would cross each bridge exactly once? As Euler showed in the article there is no way to make the walk as desired.

We come to the question: Why Euler's walk? The simple walk that goes through every connection exactly once is called the Euler's walk because it was Euler who resolved the problem of the seven bridges of Königsberg. On Figure 10 we see how Königsberg looked like at the time. As we see the four parts of the city (northern, southern and the two islands) were connected with seven bridges. There were two bridges from the smaller island to the northern and to the southern part of the city. The bigger island had one bridge to the northern and one to the southern part and there was also one bridge that connected the two islands.

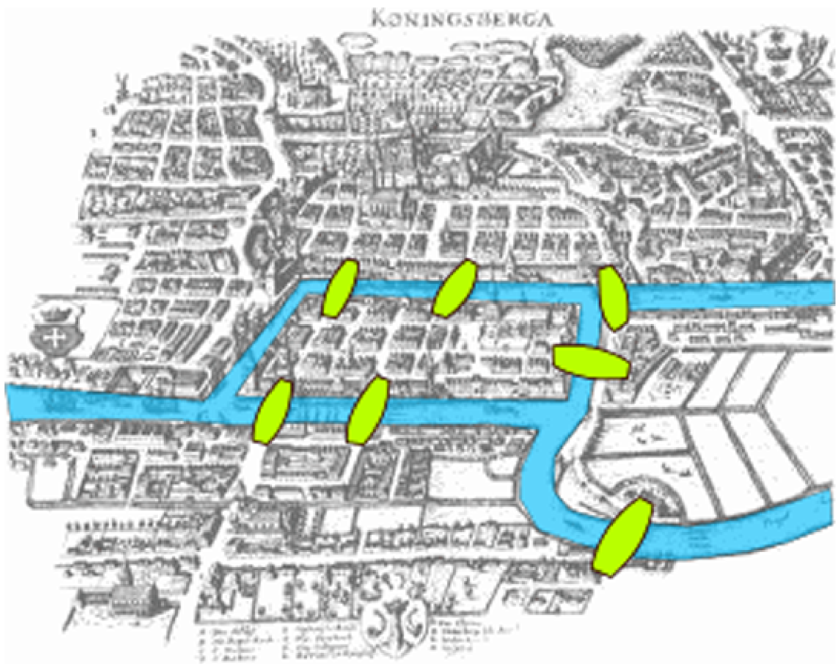

Figure 10: Four paths of the city and the seven Konigsberg bridges.

When studying the problem Euler came to the ingenious idea to mark separate parts of the city with vertices and the bridges as the connections between them. This is how he constructed a graph with four vertices and seven connections, as we can see on the Figure 11.

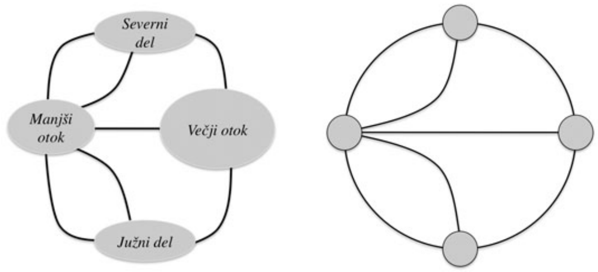

Figure 11: The model of Königsberg and bridges presented in a graph theory.

That way he modeled Königsberg and its bridges with the help of the graph theory (without knowing that at the time). In the given graph we can see the so called Euler walk. In other words - we can walk through every connection exactly once and return to the starting point. With detailed studying of the problem Euler determined that there was no solution to the problem. Several centuries later we know that the Euler walk is possible to be found only in the graph with all vertices on the even degree. Therefore such graphs with all vertices on the even degree are called the Euler graphs.

3 Summary

As we mentioned at the beginning, the graph theory is a very adequate tool for resolving logistical problems. Let us highlight some of the problems that are resolved through the graph theory and are applicable for modeling of some problems in logistics which are appearing in everyday life:

The Chinese postman problem is an example in which we are trying to search for a walk so that we go through every connection in the graph only once and do so in the shortest possible way, using the direct or undirect graph. For better understanding we could imagine a postman who is walking the streets (in our case the graph) and wants to deliver the mail for each house (vertice on the given graph) in the shortest time possible and then return to the post office (starting point). The postman is trying to save time, effort and money by finishing his job using the shortest route.

The traveling salesman problem is very similar to the Chinese postman problem at the first sight. It considers the case in which we want search for a walk using direct or undirect graph in the way to cross every vertex of the graph at least once using the shortest possible way. The salesman has to visit all the vertices in the way that he uses the shortest path (the sum of all connections used must be minimal) and return to the starting point. We can imagine that a salesman starts at point a. If the distances between every pair of points are known the question is; what is the shortest route the salesman could take to visit all the other points and return to point a?

Search for the minimum spanning tree considers the case in which we want components of unconnected graphs to be connected by using only some given edges.

Search for the shortest path comes to use when we want to find the distance or the shortest way between two vertices in weighted graph.

Finally we could state that the problems mentioned above show us the problems of the real world very nicely. The solutions of the problems of the graph theory are also very likely to show us the solutions of logistical problems in everyday life. For example:

- The paths of the snowplows can be modeled with the help of the graph theory. For this purpose we usually use one of the variations of the Chinese postman problem.
- The construction of cable or electricity network, water supply lines etc. can be resolved with the search of minimum spanning tree.
- The routes and order of transporting goods from warehouses to shops can be modeled with the merchant problem.
- The planning of the phone cable network that is connecting several different objects is modeled with the search of minimum spanning tree.
- Searching for the shortest route is already one of the common problems in everyday life. The popular GPS technology is seen on many motor vehicles as a method of searching for the easiest way to determine the right path to the chosen point on the map.

Solutions obtained through the graph theory are very helpful to the people who are resolving logistical problems. In fact, access to the relevant data for resolving problems would not be possible without such methods. According to one of the many definitions, logistics task is among other things to ensure sufficient amounts of goods are adjusted to recipients and are in the right place at the right time. There are many practical real-life logistics problems (military logistics, logistics management, business logistics, production logistics and others) which can be solved by graph theory algorithms. For example, a practical example of an application of the Chinese Postman Problem is planning of bus routing. In order to save the cost on the fuel, the bus company can model the bus stop as the vertex and the road as the edge in the bus route, then using the graph theory to obtain the optimal route that can meet the target of using the minimum of fuel but crossing every road at least once. Other applications include trash collection, road sweeping, snow-plowing, highway lawnmowers, transmission line inspections, school bus routing, etc.

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