

Poset-Game Periodicity

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1 Summary

Poset games are a family of two-player combinatorial games, including many important games that have been studied, separately, for years. In this paper, we tie these games together into a unified field by creating and proving the Poset Game Periodicity Theorem, the first major theorem applying to all poset games. This theorem states that as a poset expands along two chains, periodic patterns emerge in the winning and losing positions of the associated poset games.

2 Introduction

2.1 Posets and Poset Games

A partially-ordered set (poset) is a set X and a partial ordering of its elements — some elements are smaller than other elements, but not every pair of elements can necessarily be compared. Specifically, if $<$ is the relation, then for $a, b, c \in X$, the following must hold: if $a < b$ then $b \not< a$, and if $a < b$ and $b < c$ then $a < c$. We say that $a > b$ if and only if $b < a$; that $a \leq b$ if and only if $a < b$ or $a = b$; that $a \geq b$ if and only if $a > b$ or $a = b$; and that $a \parallel b$ (read: a is incomparable to b) if and only if neither $a \leq b$ nor $a \geq b$. A *chain* in a poset is a subset containing no pair of incomparable elements. For convenience later on, we will use the following notation to compare elements of X to subsets of X : for $a \in X$ and $S \subseteq X$, $a < S$ if and only if $a < x$ for each $x \in S$; $a > S$ and $a \parallel S$ are defined similarly. Examples of posets include the set of real numbers under their usual ordering, and the set of positive integers, with $n_1 \leq n_2$ if and only if n_1 divides n_2 . In the latter example, the powers of a prime p form a chain.

In order to play a poset game, given any poset A , two players take turns making moves. On each move, a player picks an element $x \in A$, and removes all the elements of A greater than or equal to x from the set A , to form a smaller poset A' . This becomes the new A , then the other player picks an element, and so on. The player unable to move (when $A = \emptyset$) loses. For the purposes of this paper, we will only consider poset games on finite posets.

2.2 Poset Game Background

Over the past century, many types of poset games have been named and studied. All poset game literature and research has been directed at the study of these specific games, each in isolation from the others — before this paper, no major theorem was known that applied to all poset games. The studies of these specific games have met with mixed success.

For a few specific poset games, efficient (polynomial-time) winning strategies have been found. The only such non-trivial games are games on N-free posets (that is, posets with no four elements a, b, c, d satisfying $a \parallel b, a < c, a < d, b \parallel c, b < d$, and $c \parallel d$). Such games include Nim [1], Von Neumann’s Hackendot [2], and impartial Hackenbush on trees [3]. A general strategy for poset games on N-free posets is given in [4]. Corollary 3 of this paper will offer an efficient winning strategy for another class of poset games, which need not be N-free.

However, several well-known poset games have remained unsolved for as many as fifty years, with neither a known polynomial-time winning strategy, nor a demonstration that none exists. The results offered by this paper could jump-start progress in either of those directions.

One such unsolved poset game is called Chomp, proposed in 1974 by D. Gale [5], and named later by M. Gardner [6]. An $m \times n$ bar of chocolate is divided into unit squares, and the top-left square is poisoned. On each turn, a player bites off a square, along with all the squares directly below it, directly to the right of it, and below and to the right of it. Eventually, one player is forced to eat the poisoned square, thus losing the game. If we remove the poisoned square from the set and say that one chocolate square is greater than or equal to another if the former is below and/or to the right of the latter, then we see that this is in fact a disguised poset game.

Two other well-known unsolved poset games are Schuh’s Game of Divisors and Subset Takeaway (also called the superset game). The former starts with the poset consisting of the positive divisors of a fixed integer n , excluding 1, partially ordered by divisibility, and was

proposed in 1952 by F. Schuh [7]. The latter starts with the poset consisting of all subsets of a given set, excluding the null set, partially ordered by set inclusion, and was proposed in 1982 by D. Gale [8]. Interestingly, both Subset Takeaway and Chomp are isomorphic to special cases of Schuh's Game of Divisors, where the integer n is square-free or the product of at most two distinct primes, respectively.

In Chomp, two special cases of the Poset Game Periodicity Theorem have already been stated as conjectures. Based on previous work by D. Zeilberger [9], X. Sun wrote a Maple program that calculated losing positions in Chomp by searching for periodic patterns with the top two rows [10]. The success of that algorithm led him to conjecture that, when all but the top two rows are fixed, the difference between the top two rows in losing positions is eventually periodic with respect to the length of the second row. A. Brouwer [11] verified that conjecture by computer in the special case of 3-rowed Chomp positions with at most 90,000 squares in the bottom row. Later, X. Sun wrote another Maple program, which calculated g -values for Chomp positions [12]. In the data generated, he found a periodic relation between the length of the top row and the g -value, when all other rows are fixed, and conjectured that this always holds. We will see that both of these conjectures are simple corollaries of the Poset Game Periodicity Theorem, proposed and proved in this paper.

Poset games are a branch of combinatorial game theory (that is, the study of games with no chance or hidden information). Combinatorial game theory has applications in fields such as complexity theory, artificial intelligence, error-correcting codes, algorithms, surreal analysis, and, of course, human recreation (chess, for example).

2.3 Some Game Theory Background

Since poset games are finite, impartial, combinatorial games (that is, any game will end in a finite number of moves; in any given position, both players can make the same set of moves; and there is no chance or hidden information), poset games fall under the Sprague-Grundy Theory of Games. The essential property of a position is its g -value (also called Grundy-value, nim-value, or Sprague-Grundy function). A position in a game has g -value 0 if and only if

it is a losing position (a position where the player who made the previous move can always win the game). Hence, a classification of such positions provides the basis for a winning strategy. More generally, when we are playing a disjoint sum of several games (a two-player game where, on each move, a player makes a move in exactly one of the component games, or loses if he cannot move), the winning strategy can be easily constructed if the g-values for each component game can be calculated.

Let the *mex* (“minimal excluded value”) of any set be the smallest nonnegative integer not in the set. The g-value of any position is recursively defined as the mex of the set of g-values of all game-positions that can remain after exactly one move. Thus, g-values have two properties: First, if we start from a position with g-value k , then, for any integer n , $0 \leq n \leq k - 1$, there is some move that leaves a position with g-value n . Second, there is no move from a position with g-value k to another with g-value k . (For a more complete explanation of g-values, see [3].) For any finite poset P , let $g(P)$ be the g-value of the position P in a poset game.

3 Statement of the Periodicity Theorem

In an infinite poset X , suppose we have two infinite chains C ($c_1 < c_2 < \dots$) and D ($d_1 < d_2 < \dots$), and a finite subset A , all pairwise disjoint, and assume that no element of C is less than an element of D (Figure 1 displays an example). Let

$$A_{m,n} = A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n+1}\}$$

(that is, $A_{m,n}$ is the position that results from starting with the poset $A \cup C \cup D$, then making the two moves c_{m+1} and d_{n+1}). Let k be a nonnegative integer. Then the Poset Game Periodicity Theorem states that either (1) there are only finitely many different $A_{m,n}$ with g-value k , or (2) we can find a positive integer p such that, for large enough n , $g(A_{m,n}) = k$ if and only if $g(A_{m+p,n+p}) = k$. Thus, as the poset A expands along the chains C and D , positions with any fixed g-value have a regular structure.

The paper will proceed as follows: in Section 4, we define terms and prove nine rather

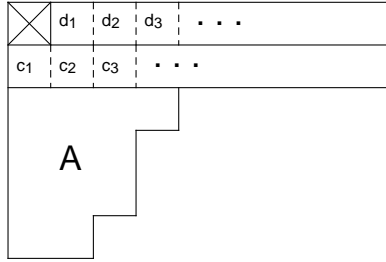


Figure 1: An example of a poset game (Chomp) to which the Periodicity Theorem applies technical lemmas; in Section 5, we use these tools to prove the periodicity theorem; in Section 6, we provide some important corollaries and implications of the theorem; and in Section 7, we suggest possible avenues for future work.

4 Preliminaries for the Periodicity Theorem

For the rest of the paper, we will assume the following:

$$\begin{aligned}
 & (X, <) \text{ is an infinite poset, containing two infinite chains } C \text{ and } D, \text{ and a} \\
 & \text{finite subset } A, \text{ all pairwise disjoint; } C = \{c_1, c_2, \dots\} \text{ with } c_1 < c_2 < \dots; \\
 & D = \{d_1, d_2, \dots\} \text{ with } d_1 < d_2 < \dots; \text{ and if } c \in C \text{ and } d \in D, \text{ then } c \not\leq d. \quad (1)
 \end{aligned}$$

For $m, n \in \mathbb{N}_0$, let

$$A_{m,n} = A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n+1}\}$$

and let

$$Q(A) = \{k \in \mathbb{N}_0 \mid \text{only finitely many positions of the form } A_{m,n} \text{ have g-value } k\}.$$

Note that this is not the same as saying that there are only finitely many solutions (m, n) to $g(A_{m,n}) = k$. For example, if $c_2 > d_2$, then, as we will see, $A_{1,1}, A_{2,1}, A_{3,1}$, etc. are all the same position.

Lemma 1. *Suppose $(m_1, n_1) \neq (m_2, n_2)$. Then $A_{m_1, n_1} = A_{m_2, n_2}$ if and only if $n_1 = n_2$, $c_{m_1+1} > d_{n_1+1}$, and $c_{m_2+1} > d_{n_1+1}$.*

Proof. Suppose $(m_1, n_1) \neq (m_2, n_2)$ and $A_{m_1, n_1} = A_{m_2, n_2}$. Since no element of D is greater

than an element of C , $A_{m,n} \cap D = \emptyset$ if and only if $n = 0$, and $\max(A_{m,n} \cap D) = d_i$ if and only if $i = n$. Hence, we can recover n from the position $A_{m,n}$, so if $A_{m_1,n_1} = A_{m_2,n_2}$ then $n_1 = n_2 = n$. Since $(m_1, n) \neq (m_2, n)$, $m_1 \neq m_2$, so without loss of generality, assume $m_1 > m_2$. If $c_{m_2+1} \not\geq d_{n+1}$ then c_{m_2+1} would be in $A_{m_1,n}$ but not $A_{m_2,n}$, contradicting the fact that they are equal. Hence, $c_{m_2+1} > d_{n+1}$ and $c_{m_1+1} > c_{m_2+1} > d_{n+1}$. Conversely, suppose $n_1 = n_2 = n$, $c_{m_1+1} > d_{n+1}$, and $c_{m_2+1} > d_{n+1}$. Then

$$A_{m_1,n_1} = A_{m_1,n} = A \cup C \cup D - \{x \in X \mid x \geq d_{n+1}\} = A_{m_2,n} = A_{m_2,n_2}. \quad \square$$

Lemma 2. *If $n_1 \neq n_2$, then $g(A_{m,n_1}) \neq g(A_{m,n_2})$.*

Proof. Without loss of generality, assume $n_1 < n_2$. Since $d_{n_1+1} \in A_{m,n_2}$, if we start with A_{m,n_2} and make the move d_{n_1+1} , the resulting position is

$$\begin{aligned} & A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n_2+1}\} - \{x \in X \mid x \geq d_{n_1+1}\} \\ &= A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n_1+1}\} \\ &= A_{m,n_1}. \end{aligned}$$

Since we can get from A_{m,n_2} to A_{m,n_1} in one move, their g-values must differ. \square

Lemma 3. *If $c_{m_1+1} \not\geq d_{n+1}$ and $m_1 \neq m_2$, then $g(A_{m_1,n}) \neq g(A_{m_2,n})$.*

Proof. This proof is similar to the proof of Lemma 2. If $m_2 < m_1$, then, since $c_{m_2+1} < c_{m_1+1}$, $c_{m_2+1} \not\geq d_{n+1}$ also. Hence, without loss of generality, assume $m_1 < m_2$. Since $c_{m_1+1} \not\geq d_{n+1}$, $c_{m_1+1} \in A_{m_2,n}$. Thus, if we start with $A_{m_2,n}$, we can make the move c_{m_1+1} and get $A_{m_1,n}$. Since there is a move from $A_{m_2,n}$ to $A_{m_1,n}$, their g-values must differ. \square

Lemma 4. *$k \in Q(A)$ if and only if one of the following is true: (i) There exists $m, n \in \mathbb{N}_0$ such that $c_{m+1} > d_{n+1}$ and $g(A_{m,n}) = k$, or (ii) there exists an $a \in A$, $m, n \in \mathbb{N}_0$ with $a < c_{m+1}$, $a < d_{n+1}$, and $g(A_{m,n} - \{x \in X \mid x \geq a\}) = k$.*

Proof. First, suppose $c_{m+1} > d_{n+1}$ and $g(A_{m,n}) = k$. Let $P = A_{m,n}$. By Lemma 1, for any $i \geq m$, $P = A_{i,n}$. By Lemma 2, for each $i \geq m$, $P = A_{i,n}$ is the only position $A_{i,y}$ that has

g-value k . Also by Lemma 2, there are at most m positions of the form $A_{i,y}$ with g-value k for $0 \leq i \leq m-1$ (at most one for each such i). Thus, in total, there are at most $m+1$ different positions of the form $A_{x,y}$ with g-value k , which is finite, so $k \in Q(A)$.

Second, suppose (ii) holds. If we make the move a from the position $A_{m',n'}$ for $m' > m$ and $n' > n$, we will get a position with g-value k . Hence, if $m' > m$ and $n' > n$, then $g(A_{m',n'}) \neq k$. For each $m' \leq m$, by Lemma 2, there is at most one position $A_{m',n'}$ with g-value k . For each $n' \leq n$, by Lemmas 1 and 3, there is at most one position of the form $A_{m',n'}$ with g-value k . Hence, there are at most $m+n+2$ positions of the form $A_{m',n'}$ with g-value k , so $k \in Q(A)$.

Finally, suppose that $k \in Q(A)$. Since any $A_{m,n}$ is finite, we can find an M and N such that, if $c_M \in A_{m,n}$ or if $d_N \in A_{m,n}$, then $g(A_{m,n}) \neq k$. By Lemma 2, we can find a $y > N$ such that $g(A_{M,y}) > k$. Let z be a move that takes $A_{M,y}$ to a position with g-value k . Since d_N and c_M are not in the resulting position, $z < d_N$, and either $d_{y+1} < c_M$ or $z < c_M$. Since $d_N < d_{y+1}$, we get $z < d_N$ and $z < c_M$. Since $z < d_N$, either $z \in D$, in which case (i) holds, or $z \in A$, in which case (ii) holds. \square

If $k \in Q(A)$, whether (i) or (ii) occurs, there is a number $T(A, k) \in \mathbb{N}_0$ such that, if $n \geq T(A, k)$, then there is a move from $A_{m,n}$ to a position with g-value k , no matter what the value of m . If (i) holds, then $c_{m'+1} > d_{n'+1}$ with $g(A_{m',n'}) = k$, and if (ii) holds, then, for some $a \in A$, $a < c_{m'+1}$, $a < d_{n'+1}$, and $g(A_{m',n'} - \{x \in X \mid x \geq a\}) = k$. For each m , by Lemma 2, there exists an n_m such that, if $n \geq n_m$, then $g(A_{m,n}) > k$. If we let $T(A, k) = \max(n_0, n_1, \dots, n_m, n' + 1)$, it will have the desired property. Now, let

$$W(A, k) = \max(\{T(B, j) \mid B \subseteq A, j \leq k, j \in Q(B)\})$$

with $\max(\emptyset)$ interpreted as 0. We will use this function later.

Lemma 5. *If $g(A_{m,n}) = k$, then $n - m \leq |A| + k$. Also, if $k \notin Q(A)$, or if $c_{m+1} \not> d_{n+1}$, then $|n - m| \leq |A| + k$.*

Proof. Suppose $g(A_{m,n}) = k$. Let

$$S = \{n' \in \mathbb{N}_0 \mid 0 \leq n' \leq n - 1, g(A_{m,n'}) > k\}.$$

For each $n' \in S$, there is at least one move from $A_{m,n'}$ to a position with g-value k . Pick one such move, and call that move $f(n')$. If $f(n') < d_{n'+1}$, then we could have made the move $f(n')$ from $A_{m,n}$ directly, which would be a move from a position with g-value k to another with g-value k , which is impossible. Hence,

$$f(n') \not\prec d_{n'+1}. \quad (2)$$

Suppose that $f(n'_1) = f(n'_2) = x$, for $n'_1 > n'_2$. Since $d_{n'_2+1} \not\prec x$ by (2), we can start with A_{m,n'_1} and make the move x , leaving a position with g-value k , then make the move $d_{n'_2+1}$, leaving another position with g-value k , a contradiction. Hence, f is injective. From (2), $f(n') \notin D$, so $f(n') \in A \cup \{c_1, \dots, c_m\}$. Since f is injective, $|S| \leq |A| + m$. By Lemma 2, at most k values of $n' < n$ satisfy $g(A_{m,n'}) \leq k$. Hence, $|S| \geq n - k$, so $n - m \leq |A| + k$.

If $k \notin Q(A)$, then, by Lemma 4, $c_{m+1} \not\prec d_{n+1}$. Thus, to finish the proof, we only need to show that, if $c_{m+1} \not\prec d_{n+1}$, then $m - n \leq |A| + k$. The proof runs exactly like the one above, but switching the roles of C and D , and using Lemma 3 instead of Lemma 2. \square

We will now add another assumption to (1), which will apply to the rest of this section:

$$\text{For each } a \in A, \text{ either } a < C \text{ or } a \parallel C, \text{ and either } a < D \text{ or } a \parallel D. \quad (3)$$

Lemma 6. *Assume (3). If $k \notin Q(A)$, and if $m \geq |A| + k$, then there exists a unique $n \in \mathbb{N}_0$ such that $g(A_{m,n}) = k$.*

Proof. Suppose that $k \notin Q(A)$ and $m \geq |A| + k$. By Lemma 5, $g(A_{m,m+|A|+k+1}) > k$. Hence, there is a move $z \in A_{m,m+|A|+k+1}$ that takes $A_{m,m+|A|+k+1}$ to a position with g-value k . We have six cases:

- $z \in A, z < C, z < D$: This is impossible, by Lemma 4, since $k \notin Q(A)$.
- $z \in A, z < C, z \parallel D$: This is impossible, since the resulting position is $B_{0,m+|A|+k+1}$, for some $B \subset A$. By Lemma 5, this has g-value greater than k .

- $z \in A, z \parallel C, z < D$: This is impossible, since the resulting position is $B_{m,0}$, for some $B \subset A$. Since $c_{m+1} \parallel z < d_1, c_{m+1} \not\parallel d_1$, so, by Lemma 5, since $m \geq |A| + k > |B| + k$, $g(B_{m,0}) > k$.
- $z \in A, z \parallel C, z \parallel D$: This is impossible, since the resulting position is $B_{m,m+|A|+k+1}$, for some $B \subset A$. By Lemma 5, this has g-value greater than k .
- $z \in C$: This is impossible, since the resulting position is $A_{m',m+|A|+k+1}$, for some $0 \leq m' \leq m - 1$. By Lemma 5, this has g-value greater than k .
- $z \in D$: In this case, $z = d_{n+1}$ for some $n \in \mathbb{N}_0$, so $g(A_{m,n}) = k$.

By process of elimination, the last case must occur, so, for any $m \geq |A| + k$, there exists an $n \in \mathbb{N}_0$ such that $g(A_{m,n}) = k$. Uniqueness follows from Lemma 2. \square

Let $f_{A,k}(m)$ be the unique (by Lemma 2) value satisfying $g(A_{m,f_{A,k}(m)}) = k$, whenever it exists. We will abuse notation slightly in order to handle the case that $f_{A,k}(m)$ does not exist. If $\alpha(B, j, m)$ is some set of conditions on B, j , and m , we will let

$$\{f_{B,j}(m) \mid \alpha(B, j, m)\} = \{i \mid \exists B, j, m \text{ with } \alpha(B, j, m) \text{ and } f_{B,j}(m) = i\}.$$

This allows the set on the left to still be defined, even if $f_{B,j}(m)$ does not always exist.

Finally, let $T = \{a \in A \mid a \parallel C, a \parallel D\}$, and let

$$H = \{A - \{x \in A \mid x \geq a\} \mid a \in T\}.$$

Lemma 7. *Assume (3). For $j \in \mathbb{N}_0, j \notin Q(A), m \geq |A| + j$, and $n \geq \max(|A| + j, W(A, j))$, there is a move from $A_{m,n}$ to a position with g-value j if and only if*

$$n \in \{f_{A,j}(i) \mid 0 \leq i \leq m - 1\} \cup \{f_{B,j}(m) \mid B \in H, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\}. \quad (4)$$

Proof. From $A_{m,n}$, for $m \geq |A| + j$ and $n \geq \max(|A| + j, W(A, j))$, suppose we make move z . Then z will satisfy exactly one of the following: (I) $z \in A$ and $z < C, z < D$; (II) $z \in A$ and $z < C, z \parallel D$; (III) $z \in A$ and $z \parallel C, z < D$; (IV) $z \in A$ and $z \parallel C, z \parallel D$; (V) $z \in C$; or (VI) $z \in D$.

After a move of type (I), by Lemma 4, since $j \notin Q(A)$, we can never be in a position with g-value j .

After a move of type (II), $A_{m,n}$ becomes $B_{0,n}$, for some $B \subset A$. Since $n \geq |A|+j > |B|+j$, Lemma 5 gives us $g(B_{0,n}) \neq j$.

After a move of type (III), $A_{m,n}$ becomes $B_{m,0}$, for some $B \subset A$. Since $z \parallel c_{m+1}$ and $z < d_1$, $c_{m+1} \not\prec d_1$. Since $m \geq |A|+j > |B|+j$, Lemma 5 gives us $g(B_{m,0}) \neq j$.

After a move of type (IV), $A_{m,n}$ can become $B_{m,n}$, for any $B \subset A$, $B \in H$. Since $n \geq W(A, j)$, if $j \in Q(B)$ then $g(B_{m,n}) \neq j$. Hence, we need only look at sets B with $j \notin Q(B)$. Thus, there is a move of type (IV) from $A_{m,n}$ that leaves a position with g-value j if and only if $n \in \{f_{B,j}(m) \mid B \in H, j \notin Q(B)\}$.

After a move of type (V), $A_{m,n}$ can become $A_{i,n}$ for any $0 \leq i \leq m-1$. Hence, there is a move of type (V) from $A_{m,n}$ that leaves a position with g-value j if and only if $n \in \{f_{A,j}(i) \mid 0 \leq i \leq m-1\}$.

After a move of type (VI), $A_{m,n}$ can become $A_{m,i}$ for any $0 \leq i \leq n-1$. Hence, there is a move of type (VI) from $A_{m,n}$ that leaves a position with g-value j if and only if $n > f_{A,j}(m)$ ($f_{A,j}(m)$ exists by Lemma 6), or, equivalently, $n \in \{i \mid i > f_{A,j}(m)\}$.

Combining these results, we see that there is a move from $A_{m,n}$ to a position with g-value j if and only if (4) holds. \square

Lemma 8. *Assume (3). For $k \notin Q(A)$ and $m \geq |A| + k + \max(|A| + k, W(A, k))$,*

$$f_{A,k}(m) - m = \min \left(\{-|A| - k, -|A| - k + 1, \dots, |A| + k\} \cap \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \left(\{f_{A,j}(i) - m \mid m - 2|A| - 2k \leq i \leq m - 1\} \cup \{f_{B,j}(m) - m \mid B \in H, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m) - m\} \right) - \{f_{A,k}(i) - m \mid 0 \leq i \leq m - 1, f_{A,k}(i) - m \geq -|A| - k\} - \{f_{B,k}(m) - m \mid B \in H, k \notin Q(B)\} \right). \quad (5)$$

Proof. By Lemma 6, $f_{A,k}(m)$ exists, and Lemma 5 constrains its possible values to the set $\{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\}$. For each $n \in \{m - |A| - k, \dots, m + |A| + k\}$, we need to check if $A_{m,n}$ can become a position with g-value j in one move for each j with $0 \leq j \leq k - 1$, and we need to check if $A_{m,n}$ can become a position with g-value k in one move. The n which satisfies the first condition but not the second is the value of $f_{A,k}(m)$. Since $m \geq |A| + k + \max(|A| + k, W(A, k))$, $n \geq \max(|A| + k, W(A, k))$. Since $n \geq W(A, k)$, $A_{m,n}$ can, in one move, become a position with g-value j for any $j \in Q(A)$. Thus, we need only check the values of j for which $0 \leq j \leq k - 1$ and $j \notin Q(A)$. Since $n \geq \max(|A| + k, W(A, k)) \geq \max(|A| + j, W(A, j))$, we can apply Lemma 7. The values of n (in the appropriate range) such that, in one move, $A_{m,n}$ can become a position with g-value j for every $0 \leq j \leq k - 1$ are:

$$\{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \left(\{f_{A,j}(i) \mid 0 \leq i \leq m - 1\} \cup \{f_{B,j}(m) \mid B \in H, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \right) \quad (6)$$

For $j < k$ and $j \notin Q(A)$, if $i \leq m - 2|A| - 2k$, then, by Lemma 5,

$$f_{A,j}(i) \leq i + |A| + j \leq (m - 2|A| - 2k) + |A| + (k - 1) = m - |A| - k - 1,$$

so $f_{A,j}(i) \notin \{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\}$. Thus, we can add the restriction $i \geq m - 2|A| - 2k + 1$ to the set $\{f_{A,j}(i) \mid 0 \leq i \leq m - 1\}$ without affecting the set (6). Since $m - 2|A| - 2k + 1 \geq 0$, we see that the set (6) is equal to:

$$\{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \left(\{f_{A,j}(i) \mid m - 2|A| - 2k + 1 \leq i \leq m - 1\} \cup \{f_{B,j}(m) \mid B \in H, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \right) \quad (7)$$

Next, we need to eliminate the n for which there is a move from $A_{m,n}$ to a position with g-value k . Since $k \notin Q(A)$, $m \geq |A| + k$, and $n \geq \max(|A| + k, W(A, k))$, Lemma 7 says that

there is a move from $A_{m,n}$ to a position with g-value k if and only if

$$n \in \{f_{A,k}(i) \mid 0 \leq i \leq m-1\} \cup \{f_{B,k}(m) \mid B \in H, k \notin Q(B)\} \cup \{i \mid i > f_{A,k}(m)\}. \quad (8)$$

Combining (7) with (8), $f_{A,k}(m)$ is the single element of

$$\begin{aligned} & \{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \\ & \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \left(\{f_{A,j}(i) \mid m - 2|A| - 2k + 1 \leq i \leq m - 1\} \right. \\ & \left. \cup \{f_{B,j}(m) \mid B \in H, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \right) \\ & - \{f_{A,k}(i) \mid 0 \leq i \leq m - 1\} - \{f_{B,k}(m) \mid B \in H, k \notin Q(B)\} - \{i \mid i > f_{A,k}(m)\} \end{aligned}$$

(this set consists of all n such that $g(A_{m,n}) = k$, and thus has the one element $f_{A,k}(m)$).

Equivalently, $f_{A,k}(m)$ is the smallest element of

$$\begin{aligned} & \{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \\ & \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \left(\{f_{A,j}(i) \mid m - 2|A| - 2k + 1 \leq i \leq m - 1\} \right. \\ & \left. \cup \{f_{B,j}(m) \mid B \in H, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \right) \\ & - \{f_{A,k}(i) \mid 0 \leq i \leq m - 1\} - \{f_{B,k}(m) \mid B \in H, k \notin Q(B)\}. \end{aligned}$$

This set is clearly the same as

$$\begin{aligned} & \{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \\ & \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \left(\{f_{A,j}(i) \mid m - 2|A| - 2k + 1 \leq i \leq m - 1\} \right. \\ & \left. \cup \{f_{B,j}(m) \mid B \in H, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \right) \\ & - \{f_{A,k}(i) \mid 0 \leq i \leq m - 1, f_{A,k}(i) \geq m - |A| - k\} - \{f_{B,k}(m) \mid B \in H, k \notin Q(B)\}. \end{aligned}$$

Subtracting m gives equation (5). □

Lemma 9. *Assume (3). For any A and k , with $k \notin Q(A)$, there exists $N_{A,k} \in \mathbb{N}_0$, $p_{A,k} \in \mathbb{N}$*

such that, if $m \geq N_{A,k}$, then $f_{A,k}(m) - m = f_{A,k}(m + p_{A,k}) - (m + p_{A,k})$.

Proof. We will prove this by strong double-induction on $|A|$ and k —for some A and k , assume that the lemma holds for all pairs (B, j) with $B \subset A$ (in particular, for $B \in H$), $j \leq k$, and $j \notin Q(B)$, and also that it holds for all pairs (A, j) with $j < k$ and $j \notin Q(A)$. In the base case for the induction, these assumptions are vacuously true. Let

$$p = \text{lcm}(\{p_{B,j} \mid B \in H, j \leq k, j \notin Q(B)\} \cup \{p_{A,j} \mid j < k, j \notin Q(A)\})$$

with $\text{lcm}(\emptyset)$ interpreted as 1, and let

$$N = \max(\{N_{B,j} \mid B \in H, j \leq k, j \notin Q(B)\} \cup \{N_{A,j} \mid j < k, j \notin Q(A)\}) \\ + |A| + k + \max(|A| + k, W(A, k))$$

with $\max(\emptyset)$ interpreted as 0. For $m \geq N$, from Lemma 8, we have the recursion (5).

Let $S(m) = \{f_{A,k}(i) - m \mid 0 \leq i \leq m-1, f_{A,k}(i) - m \geq -|A| - k\}$. From Lemma 5, if $i \leq m-1$, then $f_{A,k}(i) - m \leq f_{A,k}(i) - i - 1 \leq |A| + k - 1$. Hence, $S(m) \subseteq \{-|A| - k, \dots, |A| + k - 1\}$. This means that there are at most $2^{2|A|+2k} = 4^{|A|+k}$ possibilities for $S(m)$. Let m_p be the smallest nonnegative residue of $m \pmod{p}$. There are clearly p possibilities for m_p , so there are at most $4^{|A|+k}p$ possible pairs $(S(m), m_p)$. By the pigeonhole principle, there are two different numbers m_1, m_2 with $N \leq m_1 < m_2 \leq N + 4^{|A|+k}p$ such that $S(m_1) = S(m_2)$ and $(m_1)_p = (m_2)_p$.

By the construction of p and N , for $m \geq N$, all the sets on the right side of equation (5) other than $S(m)$, when viewed as functions of m , repeat with period p . Since $(m_1)_p = (m_2)_p$ and $S(m_1) = S(m_2)$, equation (5) implies that $f_{A,k}(m_1) - m_1 = f_{A,k}(m_2) - m_2 = r$. We also have $S(m_1 + 1) = S(m_2 + 1)$, since they can both be computed by inserting the element r into $S(m_1) = S(m_2)$, then subtracting 1 from every element of the resulting set, then eliminating from the set any element smaller than $(-|A| - k)$. Furthermore, clearly, $(m_1 + 1)_p = (m_2 + 1)_p$. Again, equation 5 implies that $f_{A,k}(m_1 + 1) - (m_1 + 1) = f_{A,k}(m_2 + 1) - (m_2 + 1)$. Repeating this argument, we get $f_{A,k}(m_1 + i) - (m_1 + i) = f_{A,k}(m_2 + i) - (m_2 + i)$ for all $i \geq 0$. Letting

$N_{A,k} = m_1$ and $p_{A,k} = m_2 - m_1$, we get $f_{A,k}(m) - m = f_{A,k}(m + p_{A,k}) - (m + p_{A,k})$ for all $m \geq N_{A,k}$. \square

5 Periodicity Theorem

Now that we have proven Lemma 9, we will drop the assumption (3) and prove our general theorem.

Poset Game Periodicity Theorem. *Assume (1). For any $k \in \mathbb{N}_0$, either there are only finitely many positions of the form $A_{m,n}$ with g -value k , or else there exists $N \in \mathbb{N}_0$, $p \in \mathbb{N}$ such that, for $m \geq N$, $f_{A,k}(m) - m = f_{A,k}(m + p) - (m + p)$.*

Proof. Note that the theorem holds for the sets C , D , A if and only if it holds for the sets $C' = C - \{c_1, \dots, c_u\}$, $D' = D - \{d_1, \dots, d_v\}$, and $A' = A \cup \{c_1, \dots, c_u\} \cup \{d_1, \dots, d_v\}$. This occurs because, for $n \geq u$, $f_{A,k}(n) - n = (f_{A',k}(n - u) - (n - u)) + (v - u)$, so the left side is eventually periodic if and only if the right side is too; and, by Lemmas 1–3, there are only finitely many positions of the form $A_{m,n}$ with g -value k if and only if there are only finitely many positions of the form $A_{m,n}$ with g -value k and with $m > u$, $n > v$, so $Q(A) = Q(A')$. By moving elements of C and D into A , we will show that we can assume (3), so that the theorem follows from Lemma 9.

Assume that $k \notin Q(A)$. If, for some i , $c_i > D$, then by Lemmas 2 and 4, $k \in Q(A)$, a contradiction. If, for any $a \in A$, $a > C$ or $a > D$, then $a \notin A_{m,n}$ for any m, n . Hence, removing a from A will have no effect on the truth or falsity of the theorem, so we can assume that no element of A is greater than all of C or all of D . We may thus assume that every element of $A \cup C \cup D$ is greater than only a finite number of other elements of $A \cup C \cup D$.

We have four steps. In the first step, we move all of the elements of $C \cup D$ that are smaller than an element of A into A , to get the new sets $A^{(1)} \supseteq A$, $C^{(1)} \subseteq C$, $D^{(1)} \subseteq D$. For each $a \in A^{(1)}$, either $a \parallel D$, or $a < d_i$ for all sufficiently large i . Thus, in the second step,

we can move elements of $D^{(1)}$ into $A^{(1)}$ (making $A^{(2)} \supseteq A^{(1)}$, $C^{(2)} = C^{(1)}$, and $D^{(2)} \subseteq D^{(1)}$) so that, for each $a \in A^{(1)}$, $a \parallel D^{(2)}$ or $a < D^{(2)}$. By the same reasoning, in the third step, we can move elements of $C^{(2)}$ into $A^{(2)}$ so that, for each $a \in A^{(2)}$, $a < C^{(3)}$ or $a \parallel C^{(3)}$. Finally, in the fourth step, we move all elements of $D^{(3)}$ that are smaller than an element of $(C^{(2)} - C^{(3)})$ into $A^{(3)}$, to get $A^{(4)}$, $C^{(4)}$, $D^{(4)}$. Note that each of these steps moves only a finite number of elements.

From any $a \in A^{(1)}$, $a < C^{(4)}$ or $a \parallel C^{(4)}$ (from the third step), and $a < D^{(4)}$ or $a \parallel D^{(4)}$ (from the second step). For any $a \in (A^{(2)} - A^{(1)}) = (D^{(1)} - D^{(2)})$, $a < C^{(4)}$ or $a \parallel C^{(4)}$ (from the third step), and $a < D^{(4)}$. For any $a \in (A^{(3)} - A^{(2)}) = (C^{(2)} - C^{(3)})$, $a < C^{(4)}$, and $a \parallel D^{(4)}$ (from the fourth step and the fact that $c_i \not\prec d_j$). For any $a \in (A^{(4)} - A^{(3)}) = (D^{(3)} - D^{(4)})$, $a < C^{(4)}$ and $a < D^{(4)}$. Hence, the assumption (3) is satisfied by $A^{(4)}$, $C^{(4)}$, $D^{(4)}$. As noted above, replacing A, C, D by $A^{(4)}, C^{(4)}, D^{(4)}$ gives an equivalent problem, and by Lemma 9, we are done. \square

6 Consequences of the Periodicity Theorem

In the special case where $k = 0$, D is the top row in Chomp, and C is the second-to-top row in Chomp, this theorem resolves X. Sun's conjecture about the periodic behavior of P-positions in Chomp [10].

By Lemma 4 and the Poset Game Periodicity Theorem, for any A and k , either: (I) We can find an M such that $f_{A,k}(m)$ does not exist for any $m > M$; (II) $f_{A,k}(m)$ is constant for $m \geq M$; or (III) $f_{A,k}(m) - m$ is periodic for $m \geq M$, with period p .

We will say we have *solved* $f_{A,k}$ if we, in case (I), calculate $f_{A,k}(m)$ for every m where $f_{A,k}(m)$ is defined; in case (II), calculate M and $f_{A,k}(m)$ for all $m \leq M$ for which $f_{A,k}(m)$ is defined; or in case (III), calculate M, p , and $f_{A,k}(m)$ for all $m < M + p$ for which $f_{A,k}(m)$ is defined.

Lemma 10. *Given $A, C, D \subset X$ satisfying (1) and given $k \in \mathbb{N}_0$, we can solve $f_{A,k}$ in a finite amount of time.*

Proof. We will proceed by strong double-induction on $|A|$ and k . Assume (vacuously in the base case, and by the induction hypothesis otherwise) that we can solve $f_{B,j}$ in a finite amount of time for all $B \subseteq A$ and $j \leq k$ (besides when $B = A$ and $j = k$). Any solution to $f_{B,j}$ can be represented by a finite set of integers. Hence, we can test a solution to $f_{A,k}$ in a finite amount of time (by symbolically checking the recursive definition of g-value given in Section 2.3). If we systematically try the countable number of possible solutions to $f_{A,k}$ (of types (I), (II), and (III)), we will eventually find one that works, in a finite amount of time. \square

Corollary 1. *Given $A, C, D \subset X$ satisfying (1) and given $k \in \mathbb{N}_0$, we can calculate $f_{A,k}(m)$ (or show that it does not exist) in $O(\log m)$ time.*

Proof. By Lemma 10, after some finite amount of time independent of m , we can solve $f_{A,k}$. After that, if $f_{A,k}$ is type (I) or (II), $f_{A,k}(m)$ can be trivially calculated in $O(1)$ time. If $f_{A,k}$ is type (III), for large m , we can reduce $(m - M) \pmod{p}$ in $O(\log m)$ time (cf. [13]), and then calculate $f_{A,k}(m) = f_{A,k}(M + (m - M)_p)$ in $O(1)$ time. Thus, in any case, $f_{A,k}(m)$ can be calculated (or shown not to exist) in $O(\log m)$ time. \square

Corollary 2. *Given $A, C, D \subset X$ satisfying (1) and given $k \in \mathbb{N}_0$, and letting m and n vary, we can check whether $g(A_{m,n}) = k$ in $O(\log m)$ time.*

Proof. By Corollary 1, we can calculate $f_{A,k}(m)$ in $O(\log m)$ time. By Lemma 5, if it exists, $f_{A,k}(m) \leq m + |A| + k$. Hence, we can check if $n = f_{A,k}(m)$ in $O(\log(m + |A| + k)) = O(\log m)$ time. In total, then, we can check in $O(\log m)$ time. \square

Corollary 3. *Given $A, C, D \subset X$ satisfying (1), the poset game starting with $A_{m,n}$ for any $m, n \in \mathbb{N}_0$ has a polynomial-time winning strategy.*

Proof. Any game position in such a game can be written as $B_{i,j}$ for $B \subseteq A$ and $i, j \in \mathbb{N}_0$. Hence, the size of the input of a position $B_{i,j}$ is $(2^{|A|} + \log i + \log j)$.

By basic properties of g-values, in order to win, we must ensure that, after each move we make, the new position has g-value 0. Suppose we are in a position $B_{i,j}$ with $g(B_{i,j}) > 0$. We must find a winning move x , in $O(\log ij)$ time, that leaves a position with g-value 0. Before we start, we solve $f_{A',0}$ for each $A' \subseteq A$. By Lemma 10, this takes a fixed, finite amount of time. Next, we calculate $t = f_{B,0}(i)$. If $t < j$, then $x = d_{t+1}$. Otherwise, calculate $f_{A,0}(m)$ for $j - |A| \leq m \leq j + |A|$. If we get $f_{A,0}(m) = j$ and $c_{m+1} \in B_{i,j}$, then $x = c_{m+1}$. There are at most $|A|$ more possible winning moves x , and each leaves a position $A'_{i',j'}$ for $A' \subset A$, $i' \leq i$, $j' \leq j$. We check if any of these has g-value 0 to find a winning move x . By Corollary 2, we have taken $(3|A| + 2)O(\log i) = O(\log i)$ time, which is polynomial with respect to the input, so this is a polynomial-time winning strategy. \square

Corollary 4. *In an infinite poset X , suppose there is an infinite chain C ($c_1 < c_2 < \dots$), and a finite subset A disjoint from C . For $n \in \mathbb{N}_0$, let $A_n = A \cup C - \{x \in X \mid x \geq c_{n+1}\}$. For large enough n , $g(A_n) - n$ is periodic with respect to n .*

Proof. Create a new poset X' as the disjoint union of X with an infinite set $\{d_1, d_2, \dots\}$. In X' , any element of D is incomparable to any element of X , the elements of X are ordered as they were before, and the elements of D form a chain $d_1 < d_2 < \dots$. The poset game on a finite subset of X' is the disjoint sum of two games: the game with elements of D (a game of Nim), and the game with elements of $A \cup C$. We know that $g(\{d_1, \dots, d_n\}) = n$, since the g-value of a nim-heap of size n is n . Hence, by elementary properties of g-values, $g(A_m) = n$ if and only if $g(A_{m,n}) = 0$ if and only if $f_{A,0}(m) = n$, which means $g(A_m) = f_{A,0}(m)$. The corollary now follows directly from the Poset Game Periodicity Theorem. \square

Note that in the case where the chain C is the top row in Chomp, this corollary proves a conjecture stated by X. Sun in [12].

Corollary 5. *Given $A, C \subset X$, we can calculate $g(A_n)$ in $O(\log n)$ time.*

Proof. From the proof of Corollary 4, calculating $g(A_n)$ is the same as calculating $f_{A,0}(n)$ in a different poset X' . Hence, this corollary follows from Corollary 1. \square

7 Future Work

The Poset Game Periodicity Theorem creates five avenues for possible future work.

First, the realm of poset games contains several long-unsolved, well-studied games. In this paper, we have found important, direct applications of the theorem to one such game, Chomp. We expect that the theorem will have applications to the theories of other such games as well, like Subset-Takeaway [8] or Schuh's Game of Divisors [7], and also to infinite poset games such as Transfinite Chomp [14]

Second, we proved, in Corollary 2, that, given A , C , D , and k , one can check if $g(A_{m,n}) = k$ in polynomial time. A possible extension of this result, which would shed additional light on poset games, would be to show that the problem of calculating $g(A_{m,n})$ is in the complexity class NP (nondeterministic polynomial), by establishing a polynomial time limit that holds for all k at once.

Third, X. Sun's algorithm to calculate g-values in Chomp [12] could be made more efficient by implementing Corollary 4. Also, by the Poset Game Periodicity Theorem, his algorithm for calculating losing positions in Chomp [10] could be altered to calculate positions with any fixed, small g-value, since these also have periodic patterns.

Fourth, allowing n chains instead of only two adds a degree of complexity to the set of positions that result. Nevertheless, it is plausible that an analogue of the Poset Game Periodicity Theorem could hold in this case as well. If so, several poset games, like Schuh's Game of Divisors (which has been unsolved for half a century), could finally be completely solved, with a general polynomial-time winning strategy.

Finally, and most importantly, the Poset Game Periodicity Theorem shows that poset games are connected not only by common rules, but also by common structure, turning these seemingly unrelated problems into a unified field. Using this theorem as a foundation and starting point, we anticipate further study of the commonalities of poset games, extending the power and reach of this new field of combinatorial game theory.

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