GEOMETRY PROBLEM FROM LATVIA

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Abstract. We describe three solutions of a geometry problem from the latvian mathematical competitions in 1994. The first elementary solution uses homotheties. The second and the third are computer assisted in the software package Maple V.

1. The problem

In the second round of the 1994 mathematical competitions in Latvia one of the problems was the following on geometry of circles [2, p. 8].

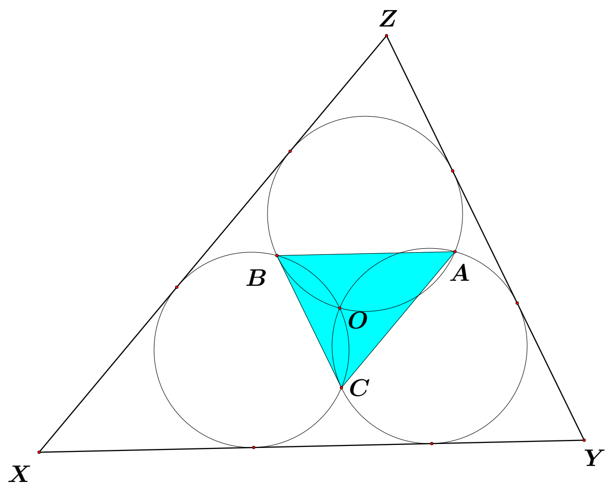

Figure 1: The area of ΔXYZ is at least nine times larger than the area of ΔABC.

Problem 1. Three equal circles intersect in the point O and also pairwise in the points A, B and C. Let T be the triangle formed by the intersections of the common tangents of these circles. Let the circles be inside the triangle T. Show that the area of T is at least nine times greater than the area of the triangle ABC.

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Let U, V and W be the centers of the circles and let X, Y and Z denote the vertices of the triangle T. The claim of the Problem 1 will be the consequence of the following theorem. In its statement the center of the nine-point circle of a triangle appears. This circle is also known as the Feuerbach circle. The most famous of its properties is that the incircle touches it from inside while the three excircles touch it from outside. The nine points from its name are the midpoints of the sides, the feet of the altitudes, and the midpoints of the segments joining the vertices with the orthocenter (the intersection of the altitudes). The readers can get more information about this circle in the books [8], [6], [5] and on the following places on the Internet:

`http://mathworld.wolfram.com/Nine-PointCircle.html`
`http://en.wikipedia.org/wiki/Nine_point_circle.html`
`http://planetmath.org/encyclopedia/NinePointCircle2.html`

Theorem 1. *(a) The triangles ABC and UVW are related by the homothety $h(F, -1)$, where F is a common center of their nine-point circles.*

(b) There exists a real number $\lambda \geq 3$ such that the triangle XYZ is the image of the triangle UVW under the homothety $h(I, \lambda)$, where I is the center of its inscribed circle.

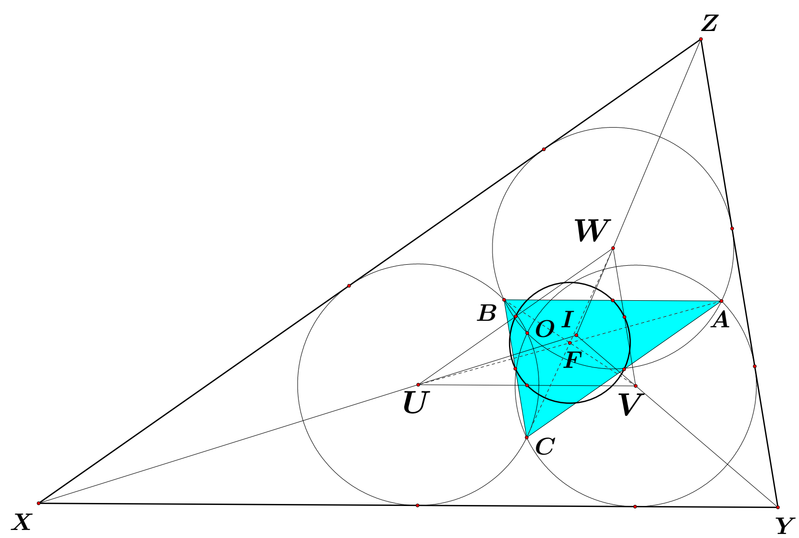

Figure 2: For the homotheties $h_1 = h(F, -1)$ and $h_2 = h(I, \lambda)$ we have $ABC = h_1(UVW)$ and $XYZ = h_2(UVW)$ with $\lambda \geq 3$.

Indeed, as a consequence of (a), we have $|UVW| = |ABC|$ (i. e. the triangles UVW and ABC have equal areas) while from (b) we get $|XYZ| = \lambda^2 |UVW|$. Hence, $\frac{|XYZ|}{|ABC|} = \lambda^2 \geq 9$, because $\lambda \geq 3$.

2. The proof of Theorem 1

Proof of Theorem 1 (a). Since the points A, B, C are the reflections of the center O of the circumcircle of the triangle UVW in its sidelines we conclude that UVW and ABC are related by the composition of the homotheties $h_3 = h(G, -\frac{1}{2})$ and $h_4 = h(O, 2)$, where G is the centroid (the intersection of medians) of UVW. This composition is in fact the homothety $h_1 = h(F, -1)$. This follows because the center F of the nine-point circle of the triangle UVW is its only fixed point and therefore is the center of the homothety and in the composition of homotheties their constants multiply. The point F is also the center of the nine-point circle of the triangle ABC because the homothety h_1 takes the complementary triangle KLM of UVW (with vertices in the midpoints of the sides of UVW) to the complementary triangle of ABC whose vertices remain on the nine-point circle of UVW since the homothety h_1 is the reflection in the point F (see Figure 3). ☐

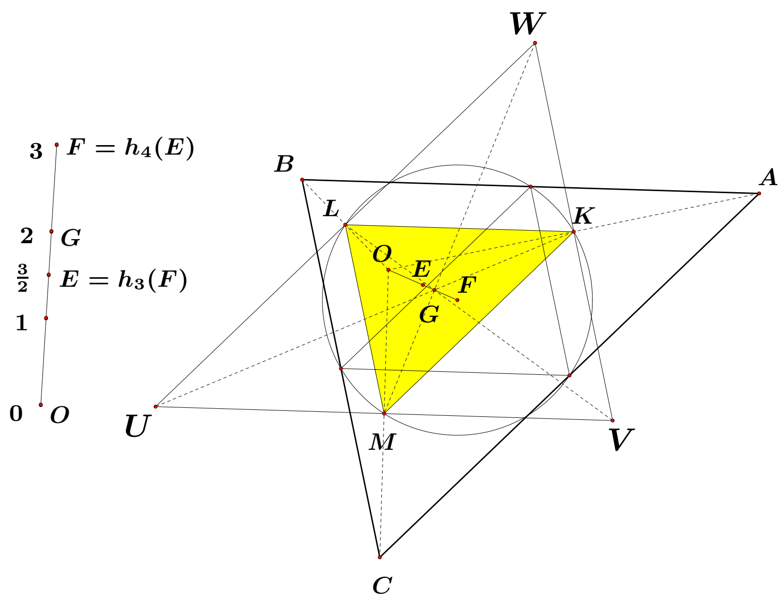

Figure 3: The composition $UVW \xrightarrow{h(G,-\frac{1}{2})} KLM \xrightarrow{h(O,2)} ABC$ is the homothety $h(F, -1)$.

Proof of Theorem 1 (b). Since the common tangent of two intersecting circles with equal radii is parallel to the line through their centers it follows that the triangles UVW and XYZ have parallel corresponding sidelines and thus are homothetic. The line XU is the bisector of the angle ZXY because the point U is at equal distance (for the length of the radius R of the circumcircle of the triangle UVW) from the rays XY and XZ. The line UX is also the bisector of the angle WUV because the corresponding sidelines of the triangles UVW and XYZ are parallel. It follows that the lines UX, VY and WZ intersect in the center I of the circle inscribed into the triangle UVW. The point I is also the center of the homothety that sends the triangle XYZ into the triangle UVW.

Its constant is equal to the ratio

$$\frac{|IX|}{|IU|} = \frac{|IU| + |UX|}{|IU|} = \frac{\frac{r}{\sin(\sphericalangle UXY)} + \frac{R}{\sin(\sphericalangle UXY)}}{\frac{r}{\sin(\sphericalangle UXY)}} = 1 + \frac{R}{r},$$

where r is the radius of the circle inscribed to the triangle UVW (see Figure 2). Since by the Euler theorem it is well known that $\frac{R}{r} \geq 2$ it follows that the constant of this homothety is at least 3. Notice that this minimal value is achieved only when the triangle UVW is equilateral. □

The above proof clearly implies that the following is also true.

Theorem 2. *(a) The triangle XYZ has at least three times larger perimeter from the perimeter of the triangle ABC.*

(b) The triangle XYZ has three times larger perimeter from the perimeter of the triangle ABC if and only if the triangles ABC, UVW and XYZ are equilateral.

3. Johnson's three circles theorem and inversion

The configuration of three circles α, β, γ having the same radius r and which intersect in the point M is mentioned earlier in [7], [3] and [4]. Johnson and Emch have shown there that if the circles intersect also in points P, Q and R then these points are on a circle ω whose radius is also equal to r (see Figure 4).

The above proof of Theorem 1 includes the proof of this result. Now we shall present its short proof with the inversion from the article [4] that describes also several other geometry problems that could be easily resolved with inversions. The inversion is the transformation of the plane that is determined by a circle. The readers can get basic information about the inversion in the books [5] and [11] and on the internet `http://mathworld.wolfram.com/Inversion.html`.

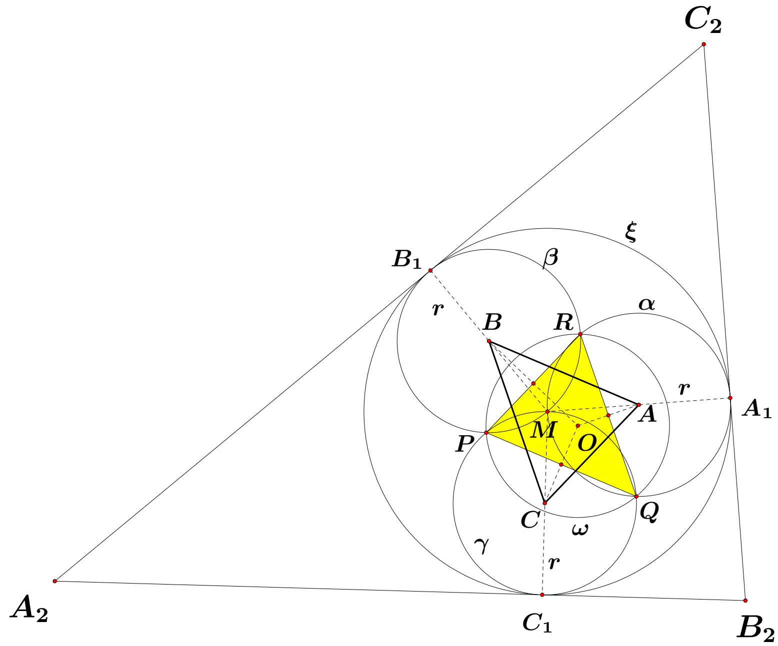

Figure 4: Johnson's three circles teorem.

Let the triangle $A_2B_2C_2$ has the inscribed circle ξ whose center is the point M and which touches the sides $|B_2C_2|$, $|C_2A_2|$, $|A_2B_2|$ in the points A_1, B_1, C_1. The images of the lines B_2C_2, C_2A_2, A_2B_2 under the inversion in the circle ξ are the circles α, β, γ which go through the point M, touch from inside the circle ξ in the points A_1, B_1, C_1, and have the same radii $r = \frac{|MA_1|}{2} = \frac{|MB_1|}{2} = \frac{|MC_1|}{2}$. Besides the point M they intersect also in the points P, Q, R that determine the circle ω (the circumcircle of the triangle PQR).

Let us now consider the reflections α', β', γ' of the circles α, β, γ in the lines QR, RP, PQ, respectively. They have the same radii r and their centers are on the perpendicular bisectors of the sides $|QR|$, $|RP|$, $|PQ|$ which intersect in the center O of the circle ω. Therefore, it follows that the circles α', β', γ' coincide with the circle ω.

Problem 2. How many times is the area of the triangle $A_2B_2C_2$ on the Figure 4 larger than the area of the triangle ABC?

By adding to the Figure 4 the triangle XYZ from the intersections of the common tangents of the circles α, β, γ on the Figure 5 from the similar triangles BMY and INY we conclude easily that ABC and XYZ are connected be a homothety whose center is the common center I of the circles inscribed into the triangles ABC and XYZ and that its ratio is $1 + \frac{r}{\varrho}$ where r i ϱ are the circumradius and the inradius of the triangle ABC and M and N are the orthogonal projections of B and I on the line XY.

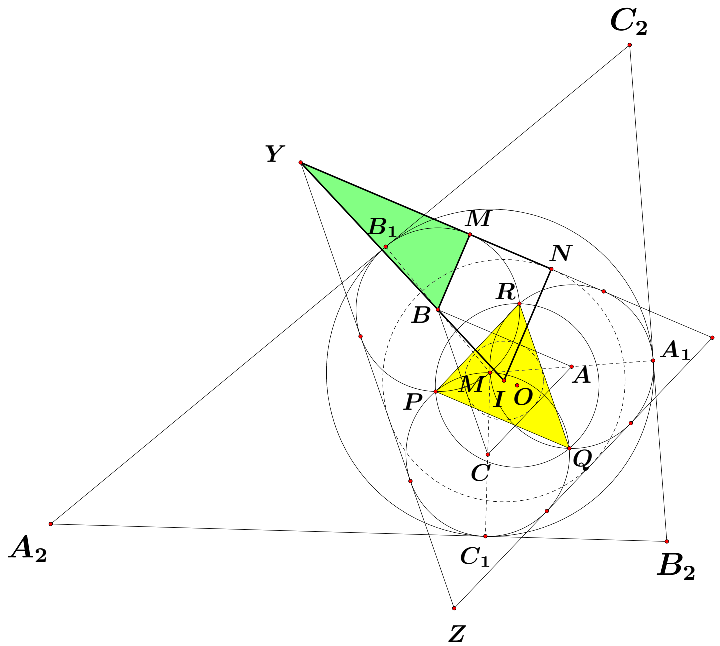

Figure 5: The triangles ABC and XYZ connects the homothety $h(I,\, 1 + \frac{r}{\varrho})$.

4. Computer assisted solution of Problem 1

Second solution of Problem 1. We shall now describe how to solve Problem 1 with the help from a computer. We use the analytic geometry of the plane and the software package Maple V. All undefined functions are explained in the article [1] whose English version is available at the author's home page: `http://www.math.hr/~cerin`.

Without loss of generality we can assume that the angles U and V of the triangle UVW are acute (i. e., smaller than $\frac{\pi}{2}$ radians) and that the vertices U, V and the center I of the inscribed circle have the coordinates $(0,\, 0)$, $(f + g,\, 0)$ and $(f,\, 1)$ for real numbers $f > 1$ and $g > 1$. Hence, the parameters f and g are the cotangents of the halves of the angles U and V while the radius of the inscribed circle is (the real number) one.

```
assume(f>1, g>1): tU:=[0, 0]: tV:=[f+g, 0]: tI:=[f, 1]:
```

The idea of our method of the proof is to determine the coordinates of the vertex W and the center O of the circumcircle. This will allow us to find the radius R of the circumcircle and then the coordinates of the vertices of the triangles XYZ and ABC. The last step is to show that the ratio of their areas is at least 9.

Let J_a, J_b, J_c be the projections of the center I of the inscribed circle onto the sidelines. Then the coordinates of J_c are $(f, 0)$ and the coordinates of J_a we get from the following system of equations.

```
tJc:=[f, 0]: tT:=[p, q]: u:=(x,y)->distance(x,y):
rJa:=solve({u(tV,tT)=u(tV,tJc), u(tI,tT)=1},{p,q});
tJa:=subs(rJa[2], tT):
```

Here the letters p and q represent the required coordinates of the point J_a. This system has two solutions. One solution are coordinates of the point J_c while the second solution are the coordinates $\frac{g^2 f+2\,g+f}{1+g^2}$ and $\frac{2\,g^2}{1+g^2}$ of J_a. In an analogous way we find the coordinates $\frac{f(f^2-1)}{f^2+1}$ and $\frac{2\,f^2}{f^2+1}$ of the point J_b.

```
rJb:=solve({u(tU,tT)=u(tU,tJc), u(tI,tT)=1},{p,q});
tJb:=subs(rJb[2],tT);
```

The point W is the intersection $UJ_b \cap VJ_a$.

```
tW:=inter(line2(tV,tJa), line2(tU,tJb));
```

The center O of the circumcircle and its radius R come from the following system of equations.

```
h:=x->u(x,tT)=R: rOR:=solve({h(tU), h(tV), h(tW)},{p, q, R});
tO:=subs(rOR,tT): vR:=subs(rOR,R):
```

Once we have the point O it is possible to get the equations of the three circles whose intersections give the points A, B, C and thus obtain these points.

```
k:=x->u(x,tT)^2-vR^2: s:=(x,y)->solve({k(x),k(y)},{p,q}):
rA:=s(tV,tW): rB:=s(tW,tU): rC:=s(tU,tV):
tA:=subs(rA[2],tT); tB:=subs(rB[2],tT); tC:=subs(rC[2],tT);
```

In order to find the coordinates of the points X, Y, Z we use six conditions from the requirements that the corresponding sidelines of UVW and XYZ are parallel and that the vertices of XYZ are at the distance

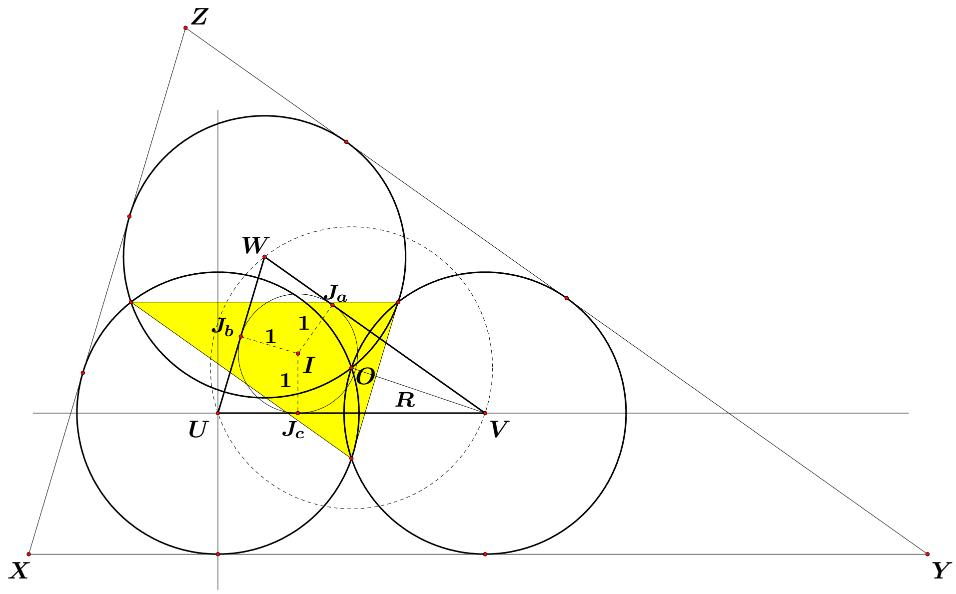

Figure 6: Determination of the coordinates of W and O and the radius R.

R from the lines UV, VW, WU.

```
tX:=[x,a]:tY:=[y,b]:tZ:=[z,c]:
j:=(x,y,z,w)->parallelQ(line2(x,y), line2(z,w)):
k:=(x,y,z)->u(x, projection(x, line2(y,z)))^2-vR^2:
rj:=solve({j(tY,tZ,tV,tW),j(tZ,tX,tW,tU), j(tX,tY,tU,tV),
     k(tX,tU,tV),k(tY,tV,tW),k(tZ,tW,tU)}, {x,a,y,b,z,c}):
tX:=subs(rj[3],tX): tY:=subs(rj[3],tY): tZ:=subs(rj[3],tZ):
```

With the coordinates of all vertices determined we can address the Problem 1 on the ratio of the areas for the triangles XYZ and ABC.

```
raz:=FS(area(tX,tY,tZ)/area(tA,tB,tC)-9);
```

The program Maple V will output that the above difference is equal to the following expression.

$$\frac{(f^2g^2 + f^2 + g^2 - 8\,f\,g + 9)\,(f^2g^2 + f^2 + g^2 + 16\,f\,g - 15)}{16\,(f\,g - 1)^2}.$$

In this way we have reduced the Problem 1 to the proof that the above quotient of polynomials is never negative.

The second parenthesis in the numerator is always positive because $f\,g > 1$ so that $16\,f\,g > 16$ and thus $f^2\,g^2 + f^2 + g^2 + 16\,f\,g - 15 > 4$.

Let us consider the first parenthesis in the numerator as a polynomial in the variable f. With the help from Maple V let us complete it to the square.

```
with(student):pz:=completesquare(op(2,raz),f):
FS(op(1,pz));FS(pz-op(1,pz));
```

Hence, the first parenthesis is equal to the sum $\frac{\left(f\,g^2+f-4\,g\right)^2}{g^2+1} + \frac{\left(g^2-3\right)^2}{g^2+1}$. This is clearly always positive except for $f = \sqrt{3}$ and $g = \sqrt{3}$ when it is zero. □

5. Another computer assisted solution of Problem 1

Third solution of Problem 1. In order to illustrate that even with computers we can have many different methods of solution for problems we shall now give another computer assisted solution of the Problem 1 which is somewhat shorter.

This time we start with the assumption that the triangle XYZ has the standard parametrization with the cotangents $f > 1$ and $g > 1$ of the halves of the angles X and Y and with the radius of the inscribed circle equal to 1.

```
assume(f>1, g>1): tX:=[0, 0]: tY:=[f+g, 0]: tI:=[f, 1]:
tZ:=[g*(f^2-1)/(f*g-1),2*f*g/(f*g-1)]:
```

Let r be the radius of the three circles in the Figure 1. Their centers U, V and W divide the segments joining the vertices with the center I of the inscribed circle (i. e. the segments $|XI|$, $|YI|$, $|ZI|$) in the ratio $r : (1 - r)$

```
o:=x->ratio2(x,tI,r,1-r): tU:=o(tX): tV:=o(tY): tW:=o(tZ):
```

The point O is at the distance r from the points U, V and W. These three conditions allow us to find the coordinates of the point O and the positive real number r which must be less than 1 because the three circles are inside the triangle XYZ.

```
tO:=[p, q]: u:=(x,y)->distance(x,y)^2-r^2:
rO:=solve({u(tU,tO), u(tV,tO), u(tW,tO), r<1},{p,q,r});
s:=x->subs(rO,x):tU:=s(tU):tV:=s(tV):tW:=s(tW):tO:=s(tO):
```

The points A, B and C are the reflections of the point O in the lines VW, WU and UV.

```
r:=(x,y,z)->reflection(x,line2(y,z)):
```

```
tA:=r(tO,tV,tW):tB:=r(tO,tU,tW):tC:=r(tO,tV,tU):
```

Finally, on the input

```
raz:=FS(area(tX,tY,tZ)/area(tA,tB,tC)-9);
```

the program Maple V will output

$$\frac{(f^2g^2+f^2+g^2-8\,f\,g+9)\,(f^2g^2+f^2+g^2+16\,f\,g-15)}{16\,(f\,g-1)^2},$$

so that we can proceed in the same way as we did in the second solution. □

6. Suggestions for research

One of the incredible attractions of mathematics is its wealth of possibilities so that even rather naive questions about a problem like ours from Latvia open up many new problems. This is called by some the mathematical daydreaming. In this section I will do a bit of mathematical daydreaming about the Problem 1 from Latvia.

The first simple idea is to increase the number of circles.

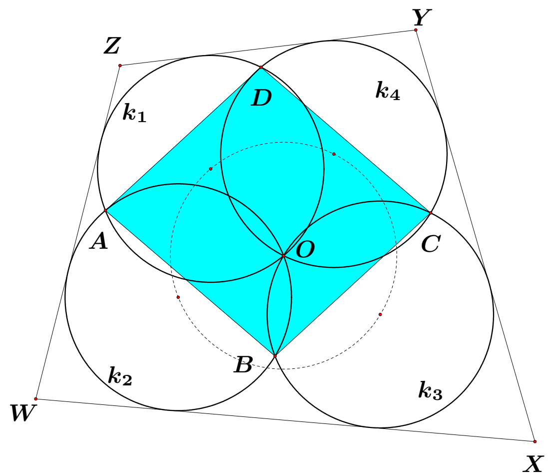

Figure 7: The case of the quadrangle.

Problem 3. Four equal circles k_1, k_2, k_3, k_4 intersect in a point O and also k_1 and k_2, k_2 and k_3, k_3 and k_4, k_4 and k_1 in the points A, B, C, D. Let T be the quadrangle from the intersections of the common tangents of these circles. Let the circles be inside the quadrangle T. Show that the area of T is at least $\frac{3+2\sqrt{2}}{2} \approx 2.9142$ times larger than the area of the quadrangle $ABCD$.

We can ask an analogous question for any finite number of circles having equal radii intersecting in a point. The real number that describes the least ratio of areas is probably connected with the corresponding regular polyhedron. As a warm up you should first try to show that $ABCD$ is a parallelogram.

The other natural possibility is to abandon the assumption that our circles have equal radii. This case can be stated as follows.

Problem 4. Let P be a point in the plane of the triangle ABC which is neither on its circumcircle nor on any of the sidelines BC, CA, AB. Let XYZ be a triangle from the common tangents of the circumcircles k_a, k_b, k_c of the triangles BCP, CAP, ABP. Let these circumcircles lie inside XYZ. Determine the lower bound for the ratio of areas of the triangles XYZ and ABC.

When the triangle ABC does not have the right angle then the circles k_a, k_b, k_c are not identical and they have equal radii if and only if the point P is the orthocenter H of the triangle ABC. When the triangle ABC is rectangular such a point does not exist. Hence, in the Theorem 1 the point O is the orthocenter (the intersection of altitudes) of the triangle ABC.

Difficulties in attempts to solve these problems or in efforts to check them in The Geometer's Sketchpad or Cabri come from the fact that two circles can have up to four common tangents. It is therefore impossible to write down the equation or to construct the tangent that is required in the problem. The expressions for the equations of all four tangents are useless because they are extremely complicated. Some new idea and a clever approach is needed here to bypass these difficulties.

In this situation we can still try to say something about the ratio of the sum Θ of areas of the circles k_a, k_b, k_c and the area S of the triangle ABC.

Problem 5. For any point P outside the sidelines BC, CA, AB the ratio $\frac{\Theta}{S}$ is at least $\frac{4\pi\sqrt{3}}{3} \approx 7.2552$.

The only thing that I can prove about the above question now is that for several important (so called central) points P of the triangle ABC (see [9]) the ratio $\frac{\Theta}{S}$ is larger than certain concrete real numbers that are somewhat smaller than $\frac{4\pi\sqrt{3}}{3}$. One such typical result is the following theorem.

Theorem 3. *If the point P is the center I of the circle inscribed into the triangle ABC, then the ratio $\frac{\Theta}{S}$ is larger than $\frac{14\pi}{9} \approx 4.8869$.*

Proof of Theorem 3. Let $A(0, 0)$, $B(f + g, 0)$, $C\left(\frac{(f^2-1)g}{fg-1}, \frac{2fg}{fg-1}\right)$ as in the proof of Theorem 1. Let us make the same assumption that the angles A and B are acute so that $f > 1$ and $g > 1$. The difference $\frac{\Theta}{S} - \frac{14\pi}{9}$ is equal to $\frac{m\,\pi}{36\,fg(f+g)(fg-1)}$, where m denotes the polynomial

$$9\,f^4g^4 + 18\,f^4g^2 - 18\,f^3g^3 + 18\,f^2g^4 - 56\,f^3g^2 - 56\,f^2g^3 + 9\,f^4 - 18\,f^3g$$
$$+54\,f^2g^2 - 18\,fg^3 + 9\,g^4 + 56\,f^2g + 56\,fg^2 + 36\,f^2 - 18\,fg + 36\,g^2 + 27.$$

By replacing f and g with $1 + u$ and $1 + v$ for some positive real numbers u and v we get the polynomial n in u and v that has only three terms with negative coefficients out of 25. Without loss of generality we can assume that $v \geq u$ so that $v = u + w$ for some real number $w \geq 0$. When in the polynomial n we replace v with $u + w$, the new polynomial has only positive coefficients. It follows that $\frac{\Theta}{S} \geq \frac{14\,\pi}{9}$ and the proof is completed. □

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